A Exam Question 1: Bose-Luttinger Liquids

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I. INTRODUCTION

Fermi liquids are at once ubiquitous and intellectually rich phases of matter. The characterizing feature of a Fermi liquid is a Fermi surface: a surface in momentum space about which there are gapless excitations. The presence of a Fermi surface is trivial in the non-interacting Fermi gas. In interacting systems, however, it instead emerges as a feature of the low-energy effective model [1].

In this paper the authors ask whether one could realize a bosonic system with gapless excitations on a surface in momentum space – essentially a bosonic analog of the Fermi liquid. There are obvious complications right off the bat, namely how such a system would realize such a surface without degeneracy pressure. Indeed, that this is not a *ubiquitous* phase of matter is clearly why it has not received attention earlier. We will put that question off, however, and seek to answer a more basic question: given a dispersion that realizes such a manifold, does the low-energy gapless theory survive in the presence of interactions? Moreover, in the case that it does, what are the properties of such a phase? Specifically, in what respects does it differ from a Fermi liquid? We'll then address potential avenues for realization at the end.

I'll start by giving some version of the punchline, which will really just serve as a roadmap for the technical details. We will find that the phase is stable in a particular parameter regime, and therefore it can be realized in principle. The regime in which it is stable is one in which the low-energy excitations have no overlap with the single-particle operators, i.e. the system is not analogous to a Fermi liquid when it comes to the existence of quasiparticles. Nevertheless, the phase is compressible and metallic. There is no analogy to Luttinger's theorem [2–4] which relates the volume of the Fermi surface to the total density. This should be evident from the fact that the Bose surface arises from the UV dispersion, and is therefore "artificial". Importantly, however, the relation between the Fermi surface and the density also implies a relation between the charge and momentum densities in a Fermi liquid (namely, they're related by the Fermi velocity). The lack of such a relationship in the bosonic system leads to a decoupling of the charge and momentum densities, which is another way of recognizing the absence of quasiparticles.

II. THE FIXED POINT

We'll now proceed to derive the 2D model. For the majority of this discussion we'll consider a Lagrangian that is invariant under the following transformations:

- U(1) charge conservation: $\psi(x) \to e^{-i\alpha}\psi(x)$
- Translation invariance: $\psi(x) \to \psi(x+\mu)$
- Particle-hole symmetry: $\psi(x) \to \psi^*(x)$

As alluded to in the above, the UV bosons are represented by a complex scalar field, $\psi(x)$. The last symmetry is worth commenting on: particle-hole symmetry enforces that the average density is equal to zero. This seems like a strange choice to make, and indeed later we'll be prompted to think about the extension to finite density. For now, however, this is a useful simplifying assumption. The UV Lagrangian is given by

$$\mathcal{L} = \psi^*\big(- u^{-1}\partial_\tau^2 - \frac{u}{4k_B^2}(-\nabla^2 - k_B^2)^2 + r\big)\psi + \frac{g}{4}|\psi|^4. \tag{1}$$

Note the absence of a ∂_τ term is due to particle-hole symmetry. It is evident that the kinetic term is minimized along a circle in momentum space of radius k_B. This will be referred to as the Bose surface. The interactions are parameterized by g, which we assume to be small, and the parameter $r < 0$ is chosen so that the amplitude of the field ϕ will acquire a finite expectation value (this is analogous to a negative mass in ϕ^4 theory). It's also worth mentioning that the Lagrangian is written in imaginary time, $\tau = -it$. We will work in imaginary time coordinates for most of this paper.

What we'll proceed to do is expand about the Bose surface. From a technical standpoint, it is useful to break the Bose surface up into discretized square patches of linear dimension 2Λ, as shown in Fig. 1. The parameter Λ is the IR cutoff, which we will vary in order to determine the renormalization group eigenvalues of the couplings. The patch procedure was originally introduced by Haldane [6] in order to bosonize a Fermi surface in dimensions greater than 1. The square dimensions are, of course, approximations that are only useful when the number of patches, $N = \frac{2\pi k_B}{2\Lambda}$, is large (otherwise one should be concerned about overlap). Within this framework we can expand the UV boson field ψ about each patch individually,

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{N}}\sum_\gamma e^{ik_B\hat{\gamma}\cdot\mathbf{x}}\psi_\gamma(\mathbf{x}), \tag{2}$$

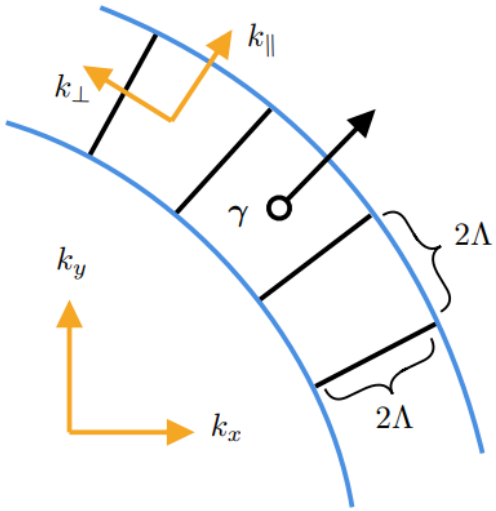


FIG. 1. Patch decomposition of the Bose surface, taken from Ref. [5]. Note the definition of $\hat{\gamma}$ as pointing radially outward.

thereby defining patch fields $\psi_\gamma(\mathbf{x})$ whose momenta are presumed to be small in comparison to the magnitude of the Bose momentum, k_B. The discretization procedure is useful because it is clean and easy to understand; the downside is that the patches themselves are not physical, which we will have to deal with at various points. In particular, we do not want any physical quantities to depend on N, just like they shouldn't depend on the IR cutoff, Λ.

What we'll now find is that, upon expanding the free part of the Lagrangian, the patch boson fields are "quasi-1D": to leading order, the dispersion only depends on the component of the momentum pointing radially outward from the Bose surface. Consider the kinetic term acting on a single patch field, ψ_γ, where we'll define $k_\parallel$ to be the component of the momentum of ψ_γ that is parallel to $\hat{\gamma}$ and $k_\perp$ to be the perpendicular component. Expanding, one finds

$$\begin{aligned}\left(e^{-ik_B\gamma\cdot\mathbf{x}}\psi_\gamma^*(x)\right)\left(\frac{1}{4k_B^2}(-\nabla^2-k_B^2)^2\right)\left(e^{ik_B\gamma\cdot\mathbf{x}}\psi_\gamma(x)\right) &\to \frac{1}{4k_B^2}(k_\parallel^2+k_\perp^2+2k_Bk_\parallel)^2\psi_\gamma^*(k)\psi_\gamma(k)\\ &\sim k_\parallel^2\ \psi_\gamma^*(k)\psi_\gamma(k).\end{aligned} \tag{3}$$

The first line is just writing out all the terms in Fourier coordinates, and then the second line discards all terms of order $\mathcal{O}(k_\parallel/k_B, k_\perp/k_B)$. One of our assumptions is that the momenta of the patch fields ψ_γ are bounded by $\Lambda \ll k_B$, so this limit is appropriate to take. Conceptually this should be thought of as linearizing about the Bose surface. We are not discarding the

curvature of the surface altogether, though: it plays a role in decreasing the phase space for scattering, as in a Fermi liquid, and its effects show up in long-distance correlation functions ($x \sim k_B/\Lambda^2$) and in the collective modes. As the renormalization group analysis will only require evaluating correlation functions at zero spacetime separation, however, we will make use of the linearized dispersion. It has been shown in analyses of bosonized Fermi liquids that this is sufficient to derive a variety of properties [7, 8].

With this in mind, we find that the linearized Lagrangian is diagonal in the patch fields and takes the form $\mathcal{L} + \mathcal{L}_I$ where

$$\mathcal{L} = \frac{1}{N} \sum_{\gamma} \psi^*_\gamma (-u^{-1}\partial_\tau^2 - u\nabla_\gamma^2 + r)\psi_\gamma \tag{4}$$

and we define $\nabla_\gamma = \hat{\gamma} \cdot \nabla$. As stated above, the curvature of the Bose surface restricts scattering processes to be in the forward scattering and BCS channels [1]. Thus $\mathcal{L}_I$ has two contributions:

$$\mathcal{L}_{\rm FS} = \frac{1}{N^2} \sum_{\gamma,\gamma'} \psi^*_\gamma \psi_\gamma g_{\rm FS}(\gamma - \gamma') \psi^*_{\gamma'} \psi_{\gamma'} \tag{5}$$

$$\mathcal{L}_{\rm BCS} = \frac{1}{N^2} \sum_{\gamma,\gamma'} \psi^*_\gamma \psi^*_{\gamma+\pi} g_{\rm BCS}(\gamma - \gamma') \psi_{\gamma'} \psi_{\gamma'+\pi}. \tag{6}$$

The couplings themselves should be rotationally symmetric and therefore depend only on the angular separation between the patches.

The next stage of analysis involves decomposing the fields into an amplitude and a phase. The reason we do this is that density-density interactions are relevant in 1D, so the diagonal components of $\mathcal{L}_{\rm FS}$ (along with our quasi-1D dispersion) should tell us that we need a new representation. The amplitude-phase decomposition is the natural choice [9]. We therefore take $\psi_\gamma = (r_0 + r_\gamma)e^{-i\phi_\gamma}$ where r_0 is a constant corresponding to the average value of the amplitude. The fields r_γ and ϕ_γ correspond to fluctuations in the amplitude and the phase, respectively. An important assumption that we will carry forward is that $\langle r_0 + r_\gamma \rangle$ is finite and independent of γ. This is another useful simplifying feature, although we will see in Sec. III that this does not have to be the case.

We now imagine that we have integrated out all momentum modes outside of the annulus of width Λ about the Bose surface. The IR limit will be defined on length scales longer than the inverse mass of the field r_γ, which by a classical expansion is easily shown to be $\sqrt{-2r}$ in terms of the UV Lagrangian. Within this limit, following the general approach originally

set forth by Haldane [10], we define a field θ_γ that is dual to the ϕ_γ fields and satisfies

$$[\phi_\gamma(\mathbf{x}), \nabla_{\gamma'}\theta_{\gamma'}(\mathbf{y})] = 2\pi i \frac{N}{k_B}\delta_{\gamma,\gamma'}\delta^{(2)}(\mathbf{x}-\mathbf{y}) \tag{7}$$

in this long-wavelength limit. We can identify the amplitude of the ψ_γ fields as $\sqrt{\rho_0 + \frac{k_B}{2\pi}\nabla_\gamma\theta_\gamma}$, thereby relating r_γ to $\nabla_\gamma\theta_\gamma$. These fields transform under the the UV symmetries as follows:

U(1):	$\phi_\gamma(x) \to \phi(x) + \alpha$	$\theta_\gamma(x) \to \theta_\gamma(x)$
Translation invariance:	$\phi(x) \to \phi(x+\mu) + k_B\hat{\gamma}\cdot\mu$	$\theta_\gamma(x) \to \theta_\gamma(x+\mu)$
Particle-hole:	$\phi_\gamma(x) \to \phi_{\gamma+\pi}(x)$	$\theta_\gamma(x) \to \theta_{\gamma+\pi}(x)$

Note that the ϕ_γ field picks up the factor from $e^{ik_B\hat{\gamma}\cdot(\mu+\mathbf{x})}$ under translation. The θ_γ field, which can be interpreted as the integrated density, does not transform under translation because the average density is equal to zero.

Now we expand the free Lagrangian defined in Eq. 4 to quadratic order in the spatial and temporal derivatives of these fields. We will work primarily in a representation of the Lagrangian in terms of only the ϕ_γ fields, which is obtained by integrating out the θ_γ fields. We're able to do this because the action is quadratic in both the ϕ_γ and θ_γ fields, allowing us to switch between representations (for more on this see Sec. C). As alluded to earlier, this paper concerns itself solely with the *form* of the resulting IR Lagrangian and not with the specific relationship between the microscopic (UV) couplings and the IR theory. It is, however, useful to consider how one might go about doing this. The IR Lagrangian takes the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_f + \mathcal{L}_I$, where

$$\mathcal{L}_0 = \frac{k_B}{4\pi N\eta}\sum_\gamma \left(v^{-1}(\partial_\tau\phi_\gamma)^2 + v(\nabla_\gamma\phi_\gamma)^2\right) \tag{8}$$

$$\mathcal{L}_f = \frac{k_B}{4\pi N^2\eta}\sum_{\gamma,\gamma'}\left(v^{-1}f_\rho^{\gamma,\gamma'}(\partial_\tau\phi_\gamma)(\partial_\tau\phi_{\gamma'}) + vf_j^{\gamma,\gamma'}(\nabla_\gamma\phi_\gamma)(\nabla_{\gamma'}\phi_{\gamma'})\right). \tag{9}$$

These are the terms that couple the charge densities $(\partial_\tau\phi_\gamma)$ and the current densities $(\nabla_\gamma\phi_\gamma)$, respectively. The Lagrangian $\mathcal{L}_0$ comes from the free part of the patch Lagrangian, Eq. (4), and $\mathcal{L}_f$ comes from the forward scattering interactions $\mathcal{L}_{\text{FS}}$ [11]. These free Lagrangians define quasi-1D Luttinger liquids with sound velocity v and the Luttinger parameter is η that live on each patch of the Bose surface. The Lagrangian $\mathcal{L}_f$ couples the charge and current densities on different patches, where we can interpret the couplings as Landau parameters (in a bosonized Fermi liquid theory, these would indeed be the Landau parameters [7, 8]).

Rotational symmetry dictates that they depend only on $\gamma - \gamma'$. These equations define the Bose-Luttinger Liquid (BLL) fixed point. Terms appearing in $\mathcal{L}_I$ are interactions that do not simply renormalize the existing phenomenological parameters, but which would induce some form of symmetry breaking towards a different state. We will discuss this in Sec. III.

At this stage, it is useful to consider the difference between this fixed point and the bosonized Fermi liquid fixed point. The first thing to note is that the parameter η is fixed to be equal to 1 in a Fermi liquid, and we will find that setting $\eta = 1$ in a BLL recovers a variety of canonical Fermi liquid results. The reason for this discrepancy is that quasiparticles are responsible for both the charge and momentum density in a Fermi liquid, and that is not the case in the BLL. To see this, it can be instructive to write the Lagrangian $\mathcal{L}_0$ in Hamiltonian form:

$$\mathcal{H}_0 = \frac{v}{4\pi N} \sum_\gamma \left(\eta (\nabla_\gamma \theta_\gamma)^2 + \frac{1}{\eta} (\nabla_\gamma \phi_\gamma)^2 \right). \tag{10}$$

Here one can consider the separate propagation of density ($\nabla_\gamma \theta_\gamma$) and current ($\nabla_\gamma \phi_\gamma$) fluctuations, with respective velocities $v\eta$ and v/η. The condition that $\eta = 1$ means that these fluctuations propagate at identical speeds, which is related to the fact that the momentum and charge densities are coupled in the Fermi liquid: the low-energy excitations, quasiparticles, are responsible for the propagation of both. One can also compare to the standard 1D Luttinger liquid theory results to find that $\eta = 1$ corresponds to the free Fermi gas in 1D, while $\eta \neq 1$ corresponds to a generic interacting Luttinger liquid without quasiparticles [9]. We can therefore think of $\eta = 1$ in the low-energy theory as indicating that it is adiabatically connected to the free Fermi gas. A related point of difference between the BLL and the Fermi liquid is that there are actually two sets of Landau parameters in the BLL, f_ρ and f_j, which couple the charge and momentum densities (respectively) on different patches. In a Fermi liquid one would find $f_\rho = f_j$ for the same reason stated above. This means that there are two zero sound modes in the BLL where (in a spinless Fermi liquid) there would otherwise be one. We will see this in Sec. IV.

III. SYMMETRY AND STABILITY

In order to discuss the stability of this fixed point, we first have to discuss the emergent symmetry group that it realizes. Perturbations to the fixed point should break this symmetry, otherwise they will simply renormalize the existing couplings. Looking at Eq. (8) in

isolation, one might think that action would be invariant under $\phi_\gamma(x) \to \phi_\gamma(x) + f_\gamma(\hat{\gamma}_\perp \cdot x)$, whereby each patch phase field is shifted by a function that can depend on the perpendicular direction in real space. As noted earlier, however, Eq. (8) is uses a linearized dispersion that does not include the curvature of the Bose surface. While we are ignoring these terms in a perturbative sense, they are not formally zero and therefore this is not the correct symmetry group. This means that the transformation cannot be a function of position, so one might propose $\phi_\gamma(x) \to \phi_\gamma(x) + f_\gamma$ where f_γ can differ from patch to patch. In the low-energy $N \to \infty$ limit, this would realize a $\mathrm{U}(1)^\infty$ symmetry group whereby charge is conserved at each point on the Bose surface. This, as well, is too large of a symmetry group. Take the Fermi liquid as an example: while a non-interacting Fermi gas in the thermodynamic limit explicitly realizes $\mathrm{U}(1)^\infty$ in the UV, it is only *approximately* realized in the interacting Fermi liquid. Similarly, the BLL appears to approximately realize the $\mathrm{U}(1)^\infty$ group, but the only U(1) conserved density is the microscopic U(1). To this end, the authors claim that the correct symmetry group is the loop group, LU(1), which corresponds to the set of transformations $\phi_\gamma(x) \to \phi_\gamma(x) + f_s(\gamma)$ where $f_s(\gamma)$ is a smooth function of γ. It is worth underlining the conceptual point – what the authors are doing is defining a symmetry group that will be well-behaved we increase the number of patches on the Bose surface. The idea is that, despite the usefulness of the patch decomposition, at the end of the day we should be able to conceptually mend these patch fields back into a single low-energy field. Transformations allowed under the $\mathrm{U}(1)^\infty$ would make for patch configurations that are singular (e.g. discontinuous) functions of γ, which are therefore not physical.

Given this, we can now discuss allowed perturbations to the fixed point. The authors claim that the most relevant perturbation is the BCS pairing interaction,

$$\frac{1}{N^2} \sum_{\gamma,\gamma'} g_{\mathrm{BCS}}(\gamma - \gamma') \cos(\phi_\gamma + \phi_{\gamma+\pi} - \phi_{\gamma'} - \phi_{\gamma'+\pi}). \tag{11}$$

This is a reasonable perturbation as it is invariant with respect to all the UV symmetries yet it breaks the emergent LU(1) symmetry down into a subgroup of functions $\tilde{f}_s(\gamma)$ that obey $\tilde{f}_s(\gamma) = -\tilde{f}_s(\gamma + \pi)$. That it is the "most relevant" will only be meaningful after the renormalization group analysis. Before proceeding, however, they note that one might also think to construct terms that are proportional to $\cos(\theta_\gamma)$, which is a common perturbation to Luttinger liquids (see Appendix E of Ref. [9]). The authors argue, however, that this is not a legal perturbation to the system because the vertex operator $e^{i\theta_\gamma}$, by virtue of the

commutation relations in Eq. (7), creates a vortex in the associated patch field ϕ_γ. Having a vortex on one patch field ϕ_γ and not on an adjacent patch field $\phi_{\gamma'}$ is a singular configuration, as one cannot pass smoothly between field configurations of different vorticity. For the same reason that the emergent symmetry group is LU(1) instead of $\mathrm{U}(1)^\infty$, then, the operators $e^{i\theta_\gamma}$ are not legal perturbations to the BLL fixed point.

The first thing we have to do for the RG analysis is compute some correlation functions. The correlation function between ϕ_γ fields is trivial if we consider just the Lagrangian $\mathcal{L}_0$, ignoring the Landau parameters:

$$G_\phi^{\gamma\gamma'}(\mathbf{k},\omega) = \langle \phi_\gamma(\mathbf{k},\omega)\phi_{\gamma'}(-\mathbf{k},\omega)\rangle = \delta_{\gamma\gamma'}\frac{2\pi v\eta l_\Lambda}{\omega^2+v^2k_\gamma^2}. \tag{12}$$

As shown in Appendix A, including the Landau parameters gives a correction of order $\mathcal{O}(1/N)$. We can therefore safely ignore their contribution to the $\phi\phi$ correlation function in the IR limit, as we will be taking $\Lambda \to 0$ and $N \to \infty$.

Now let's compute the correlator of the vertex operators, $\langle e^{i\phi_\gamma(x)}e^{-i\phi_{\gamma'}(0)}\rangle$. The first thing to do is rearrange these a bit. We'll make use of the fact that $\langle e^{i\hat{X}}\rangle = e^{-\langle \hat{X}^2\rangle/2}$ for an operator with $\langle \hat{X}\rangle = 0$:

$$\begin{aligned}\langle e^{i\phi_\gamma(x)}e^{-i\phi_{\gamma'}(0)}\rangle &= \langle e^{i(\phi_\gamma(x)-\phi_{\gamma'}(0))}\rangle \\ &= \exp\Big(-\frac{1}{2}\langle\big(\phi_\gamma(x)-\phi_{\gamma'}(0)\big)^2\rangle\Big).\end{aligned} \tag{13}$$

The argument in the exponential is now given by

$$2\pi\eta v l_\Lambda \int \frac{d^2k d\omega}{(2\pi)^3}\,\frac{\delta_{\gamma,\gamma'}e^{i(k\cdot x+\omega\tau)}-1}{\omega^2+v^2k_\gamma^2} \tag{14}$$

where we use the result from Eq. (12). The bounds on the momentum integral are $[-\Lambda,\Lambda]^2$ and the frequency integral is over $(-\infty,\infty)$. We'll find that the integral over $k_\parallel$ is formally infinite, so we'll have to add a small-momentum cutoff $1/L$ which can be interpreted as imposing a finite system size. Of course, we will only consider results that are finite as $L\to\infty$. Performing the frequency integral, we are left with

$$\int \frac{d^2k d\omega}{(2\pi)^3}\frac{e^{i(k\cdot x+\omega\tau)}-1}{\omega^2+v^2k_\gamma^2} = \frac{1}{v}\int_{-\Lambda}^{\Lambda}\frac{dk_\perp}{2\pi}\Bigg(e^{ik_\perp x_\perp}\Bigg(\int_{1/L}^{\Lambda}\frac{dk_\parallel}{2\pi}\frac{1}{k_\parallel}e^{ik_\parallel x_\parallel - |kv\tau|}\Bigg) - \Bigg(\int_{1/L}^{\Lambda}\frac{dk_\parallel}{2\pi}\frac{1}{k_\parallel}\Bigg)\Bigg). \tag{15}$$

The second integral inside the parentheses clearly just gives $\ln(\Lambda L)$. In order to simplify things a bit, we add and subtract 1 from the first integrand. This is because the integral

over the exponential minus 1 actually converges without the system size cutoff $1/L$, which makes things better-behaved as we take $L \to \infty$:

$$\begin{aligned}\int_{1/L}^{\Lambda} \frac{dk_\parallel}{2\pi}\frac{1}{k_\parallel}e^{ik_\parallel x_\parallel - |kv\tau|} &\approx \int_0^{\Lambda} \frac{dk_\parallel}{2\pi}\frac{1}{k_\parallel}\big(e^{ik_\parallel x_\parallel - |kv\tau|} - 1\big) + \int_{1/L}^{\Lambda} \frac{dk_\parallel}{2\pi}\frac{1}{k_\parallel} \\ &= -\frac{1}{2\pi}\big(\ln(\Lambda\sqrt{x_\parallel^2 + (v|\tau| + 1/\Lambda)^2}) - \ln(\Lambda L)\big) \\ &= -\frac{1}{2\pi}\ln\left(\frac{1}{L}\sqrt{x_\parallel^2 + (v|\tau| + 1/\Lambda)^2}\right)\end{aligned} \tag{16}$$

We therefore have

$$\int \frac{d^2k d\omega}{(2\pi)^3}\frac{e^{i(k\cdot x + \omega\tau)} - 1}{\omega^2 + v^2k_\gamma^2} = -\frac{1}{2\pi v}\int_{-\Lambda}^{\Lambda}\frac{dk_\perp}{2\pi}\left(\ln\left(\frac{1}{L}\sqrt{x_\parallel^2 + (v|\tau| + 1/\Lambda)^2}\right)e^{ik_\perp x_\perp} + \ln(\Lambda L)\right). \tag{17}$$

As the authors point out, it's worth considering the limits of this integral prior to exponentiating it. If we have that $x_\perp\Lambda \ll 1$ then the complex exponential will be approximately 1 for the entire range of integration and the integral over $k_\perp$ just gives a factor of l_Λ^{-1} (defined earlier). If $x_\perp\Lambda \gg 1$, then the integral over the complex exponential becomes a delta function $\delta(x_\perp)$. As the condition $x_\perp\Lambda \gg 1$ implies $x_\perp$ is finite, this means the integral over the first term gives zero and the result is proportional to $\ln(\Lambda L)$. Thus, when we remove the system size dependence by taking $L \to \infty$ at the end, we would find that the integral goes to $-\infty$, and therefore that the vertex correlator vanishes. With this in mind, the authors propose a simplified form for the vertex correlator:

$$\begin{aligned}\langle e^{i\phi_\gamma(x)}e^{-i\phi_\gamma(0)}\rangle &\sim \delta_{\gamma,\gamma'}\delta_\Lambda(x_\perp)\exp\left(-\eta\ln(\Lambda\sqrt{x_\parallel^2 + (v|\tau| + 1/\Lambda)^2})\right) \\ &= \frac{\delta_{\gamma,\gamma'}\delta_\Lambda(x_\perp)}{((\Lambda\hat{\gamma}\cdot x)^2 + (1 + \Lambda v|\tau|)^2)^{\eta/2}}\end{aligned} \tag{18}$$

where the function δ_Λ is defined as

$$\delta_\Lambda(x) = \begin{cases} 1 & |x| \le \Lambda^{-1} \\ 0 & |x| > \Lambda^{-1} \end{cases}. \tag{19}$$

We can now proceed with the renormalization group analysis. First let's define the rules of the game, as they are slightly different than the conventional approach [1]. We will integrate out modes within the annulus $s\Lambda \le \sqrt{k_\gamma^2 + \omega^2/v^2} \le \Lambda$ where s is a parameter slightly less than 1. This involves only two of the three spacetime dimensions, which is intentional because the free action S_0 does not depend on $k_\perp$. As a result, our patches will

no longer be squares in momentum space after mode elimination. Thus our next step is to increase the number of patches $N \to N' = N/s$ so that they are now square with dimension $s\Lambda \times s\Lambda$. When changing the IR cutoff, we want to be sure that we're not changing anything about the UV theory. That is to say, the UV boson field $\psi(x)$ should not rescale under the renormalization group. As the number of patches changes, then, this implies that the *patch* fields will rescale as follows:

$$\psi(x) = \frac{1}{\sqrt{N}} \sum_{\gamma} e^{ik_B \hat{\gamma}\cdot x} \psi_\gamma(x) = \frac{1}{\sqrt{sN'}} \sum_{\gamma'} e^{ik_B \hat{\gamma}'\cdot x} \left(\sqrt{s}\psi'_{\gamma'}(x)\right) \tag{20}$$

where the primed quantities are those obtained after the patch rescaling. Now let's consider the BCS pairing term in the language of these patch fields:

$$\mathcal{L}_{\text{BCS}} = \frac{1}{N^2} \sum_{\gamma,\gamma'} \psi^*_\gamma \psi^*_{\gamma+\pi} g_{\text{BCS}}(\gamma - \gamma') \psi_{\gamma'} \psi_{\gamma'+\pi}. \tag{21}$$

We can see immediately that the rescaling of the patch fields will cancel out with the rescaling of N. We can therefore ignore the rescaling of N when considering the representation in terms of the ϕ_γ and θ_γ fields (which will not rescale).

Consider that the action is generically of the form $S = \int d^3r\, \mathcal{L}$ where I take $d^3r = d^2x d\tau$. When computing the scaling dimension of some interaction under RG, one usually compares the value to the number of spacetime dimensions. For example, in $2+1$D one might find that the scaling dimension of an operator is given by the parameter χ, and therefore the relevance or irrelevance of the operator under RG is determined by seeing whether $-3+\chi$ is positive or negative. The negative sign comes from the fact that a uniform rescaling of spatial dimensions takes us from $d^3r \to s^{-3} d^3r'$. The reason for being so explicit about this is that, although we are in $2+1$ spacetime dimensions, only two of those dimensions are being rescaled under mode elimination. The remaining dimension, along $\hat{\gamma}_\perp$, is also rescaled but that rescaling is compensated by a proportionate increase in the number of patches. *We will therefore compare the scaling dimension of the operator to* -2 *as opposed to* -3*, which reflects the fact that the free Lagrangian* $\mathcal{L}_0$ *is effectively* $1+1$*-dimensional* [12].

With that out of the way, let's get to the mode elimination. We are now working with the IR fixed point, Eq. (8), and the perturbation defined in Eq. (11). As $\mathcal{L}_0$ is diagonal in Fourier space, we can define $\phi_\gamma = \phi^>_\gamma + \phi^<_\gamma$ where the former have momenta and frequencies in the annulus defined above. These are the fast modes that we will integrate out. The procedure will be to compute effective contributions to the action, obtained by expanding

the path integral and then re-exponentiating. It is convenient to represent both operations in the form of a cumulant expansion, which gives an expression for the effective action after renormalization:

$$S_{eff}[\phi^<] \approx S_0[\phi^<] + \langle \delta S[\phi^< + \phi^>] \rangle_> + \frac{1}{2}\big(\langle \delta S[\phi^< + \phi^>]^2 \rangle_> - \langle \delta S[\phi^< + \phi^>] \rangle_>^2 \big) + \dots . \quad (22)$$

The term $S_0[\phi^<]$ is just the free action after removing the $\phi^>$ modes, which requires no averaging. Notationally, $\langle \dots \rangle_>$ means averaging over the fast modes. The term δS is the contribution of the BCS pairing term to the action,

$$\delta S[\phi] = \frac{1}{N^2} \sum_{\gamma,\gamma'} g_{\rm BCS}(\gamma - \gamma') \int d^3 r \ \cos\big(\phi_\gamma(r) + \phi_{\gamma+\pi}(r) - \phi_{\gamma'}(r) - \phi_{\gamma'+\pi}(r) \big). \quad (23)$$

The tree-level RG exponent is obtained by computing the second term in Eq. (22). To do this, we expand the cosine (I use the quantity $\varphi_{\gamma,\gamma'}(r)$ to represent the sum of all terms inside the cosine):

$$\begin{aligned} \delta S[\phi^< + \phi^>] &= \frac{1}{2N^2} \sum_{\gamma,\gamma'} g_{\rm BCS}(\gamma - \gamma') \int d^3 r \Big(e^{i\varphi_{\gamma,\gamma'}(r)} + e^{-i\varphi_{\gamma,\gamma'}(r)} \Big) \\ &= \frac{1}{2N^2} \sum_{\gamma,\gamma'} g_{\rm BCS}(\gamma - \gamma') \int d^3 r \Big(e^{i\varphi^>_{\gamma,\gamma'}(r)} e^{i\varphi^<_{\gamma,\gamma'}(r)} + e^{-i\varphi^>_{\gamma,\gamma'}(r)} e^{-i\varphi^<_{\gamma,\gamma'}(r)} \Big). \end{aligned} \quad (24)$$

This latter step is legal because the Lagrangian is diagonal in Fourier space, so all these terms commute. Now we integrate out the $\phi^>$ fields, which amounts to taking the expectation value of the expression in parentheses with respect to these fields (i.e. using the action $e^{-S_0^>}$). This gives us the expectation value of the vertex operator. We then use the identity from earlier to obtain

$$\langle \delta S[\phi^< + \phi^>] \rangle_> = \frac{1}{N^2} \sum_{\gamma,\gamma'} g_{\rm BCS}(\gamma - \gamma') \int d^3 r \ \cos(\varphi^<_{\gamma,\gamma'}(r)) e^{-\frac{1}{2}\langle (\varphi^>_{\gamma,\gamma'}(r))^2 \rangle}. \quad (25)$$

The expectation value in question is straightforward. The expectation value of two ϕ_γ fields on different patches is zero, so we're left with $\langle (\varphi^>_{\gamma,\gamma'}(r))^2 \rangle = 4 \langle (\phi^>_\gamma(r))^2 \rangle$. The latter expectation value is

$$\begin{aligned} \langle (\phi^>_\gamma(r))^2 \rangle &= \int_{s\Lambda}^{\Lambda} \frac{d\omega \ dk_\parallel}{(2\pi)^2} \int_{-\Lambda}^{\Lambda} \frac{dk_\perp}{2\pi} \frac{2\pi\eta v l_\Lambda}{\omega^2 + v^2 k_\parallel^2} \\ &= 2\pi\eta \int_{s\Lambda}^{\Lambda} \frac{dq}{(2\pi)^2 |q|} \int_{-\pi}^{\pi} d\theta \\ &= -\eta \ln(s) \end{aligned} \quad (26)$$

The integration bounds on the $d\omega\ dk_{\|}$ integral mean that we're integrating the annulus defined earlier. In the second line I switch to polar coordinates, absorbing a factor of v. The integration of momenta in the perpendicular direction is trivial, giving a factor of l_Λ^{-1}. We don't change the bounds on that integral to $\pm s\Lambda$ because the equivalent operation is accomplished by reparameterizing the patches to be squares (which, it is assumed, we do after mode elimination). Inserting this result into the above expression, we find

$$\langle \delta S[\phi^< + \phi^>] \rangle_> = s^{2\eta} \frac{1}{N^2} \sum_{\gamma,\gamma'} g_{\rm BCS}(\gamma - \gamma') \int d^3r\ \cos(\varphi^<_{\gamma,\gamma'}(r)). \tag{27}$$

Now we can determine the tree-level scaling of the parameter $g_{\rm BCS}$. Again, note that from a simple rescaling of the spacetime dimensions ($x_\gamma \to x_\gamma/s$, $\omega \to \omega/s$), which is necessary to ensure that the integrals run over the same values after mode elimination, we pick up a factor of s^{-2}. We therefore find

$$\delta S_{eff} \sim s^{2\eta-2} \frac{1}{N^2} \sum_{\gamma,\gamma'} g_{\rm BCS}(\gamma - \gamma') \int d^3r'\ \cos(\varphi^<_{\gamma,\gamma'}(r')). \tag{28}$$

This, of course, looks just like the definition of δS in the first place except without any of the fast modes and multiplied by an overall factor of $s^{2\eta-2}$. Replacing $s = e^{-t}$ where t is referred to (in this paper) as the RG time, we define a new coupling $g_{\rm BCS}(t,\gamma)$ that depends on the parameter t. We are interested in the limit $t \to \infty$, which would correspond to taking $\Lambda \to 0$. We find

$$g_{\rm BCS}(t,\gamma) = g_{\rm BCS}(0,\gamma) e^{(2-2\eta)t} \implies \frac{dg_{\rm BCS}}{dt} = (2-2\eta) g_{\rm BCS} \tag{29}$$

From this we can immediately read off the RG eigenvalue, $2-2\eta$. As $t \to \infty$, the sign of this eigenvalue determines whether the perturbation will increase or decrease. If it decreases, then there is a low-energy scale at which it can be ignored altogether – it is formally "irrelevant". Thus, for $\eta > 1$, the RG eigenvalue is negative and the Bose-Luttinger liquid is stable to the BCS pairing interaction. We can also see now that additional symmetry-breaking cosine terms, which would have to contain the sum of more than four ϕ_γ fields, would be less relevant than the BCS pairing term.

It is now worth refocusing attention on the fact that, for a Fermi liquid, analogous bosonization schemes yield $\eta = 1$. Thus one would find that the BCS coupling is marginal at tree level [1]. This is not strictly true for the BLL as η is not fixed. If it were the case

that $\eta = 1$, however, one must continue the calculation to one-loop order to determine how it rescales under RG. We perform this calculation in Appendix B, finding the familiar result

$$\frac{dg^l_{\text{BCS}}}{dt} = (2 - 2\eta)g^l_{\text{BCS}} - C(g^l_{\text{BCS}})^2 \tag{30}$$

where $C > 0$ is a constant and we decomposed the angular dependence of $g_{\text{BCS}}(\gamma)$ into harmonics,

$$g^l_{\text{BCS}} = \int d\gamma \ \cos(\gamma l) g_{\text{BCS}}(\gamma). \tag{31}$$

Due to Bose symmetry we must have $g_{\text{BCS}}(\gamma) = g_{\text{BCS}}(\gamma + \pi)$, which implies that $l \in 2\mathbb{Z}$.

In the case that $\eta = 1$, like the Fermi liquid, we would therefore find that the sign of g^l_{BCS} determines its relevance: positive couplings are marginally irrelevant and negative couplings are marginally relevant. In the case that the BCS coupling is marginally relevant, the harmonic $g^{l^*}_{\text{BCS}}$ that is most negative will determine the resulting spontaneous symmetry breaking. If we define $\phi^{\pm}_{\gamma} = \phi_{\gamma} \pm \phi_{\gamma+\pi}$, the authors claim that $\langle \phi^+_{\gamma} \rangle$ would acquire a finite expectation value. This comes from the fact that the BCS pairing perturbation breaks down the LU(1) symmetry group into a subgroup of transformations characterized by smooth functions that obey $\tilde{f}_s(\gamma) = -\tilde{f}_s(\gamma + \pi)$. This will result in pairing between ϕ_{γ} and $\phi_{\gamma+\pi}$, analogous to the BCS pairing leading to superconductivity. The pairing occurs in the ϕ^+_{γ} channel because it is invariant under these transformations. As ϕ^-_{γ} is not, it will not be gapped out by pairing interactions [13]. The resulting state should still have a Bose surface that is described in the IR by the ϕ^-_{γ} fields. Formally this would be equivalent to a BLL where the UV bosons were represented by real scalar fields. As such, there would be no U(1) charge conservation (which must be the case, as ϕ^-_{γ} is not charged under the microscopic U(1)) and the real bosonic fields would obey $\Phi_{\gamma}(x)^* = \Phi_{\gamma+\pi}(x)$. The authors demonstrate that the low-energy theory for a real bosonic system takes the same form as the one for the complex bosonic theory and is stable for $\eta > 4/3$.

IV. PHENOMENOLOGY

Let's now dive into the features of the fixed point, where we'll be able to make a closer comparison to the 2D Fermi liquid. The first thing we'll look at is the specific heat. In the absence of Landau parameters we have N independent, decoupled Luttinger liquids. Each Luttinger liquid Hamiltonian is diagonalized in a basis of massless bosons with dispersion

$\omega = vk_\gamma$. Consider one such patch. The low-energy Hamiltonian is of the form

$$\mathcal{H}_\gamma = v\frac{1}{N_k}\sum_{k_\gamma} |k_\gamma| a^\dagger_{k\gamma} a_{k\gamma} \to v\int_{-\infty}^{\infty} \frac{dk_\gamma}{2\pi} |k_\gamma| a^\dagger_{k\gamma} a_{k\gamma} \tag{32}$$

where $a_{k\gamma}$ are bosons on patch γ and we took the continuum limit. The specific heat of this system is given by $C = \partial\langle H_\gamma\rangle/\partial T$. We can easily write the expectation value at fixed T using $\langle n_k\rangle = n_B(T)$ where n_B is the Bose-Einstein distribution function. We find

$$\langle \mathcal{H}_\gamma\rangle = \frac{v}{2\pi}\int_{-\infty}^{\infty} \frac{|k|dk}{e^{\beta v|k|}-1} = \frac{T^2}{\pi v}\int_0^\infty \frac{x}{e^x-1} = \frac{T^2}{\pi v}\xi(2) \tag{33}$$

where ξ is the Riemann-Zeta function and $\xi(2) = \pi^2/6$. The specific heat due to a single patch is therefore

$$C_{1D} = \frac{\partial\langle \mathcal{H}_\gamma\rangle}{\partial T} = \frac{\pi T}{3v}. \tag{34}$$

Given this fact, we should interpret the 1D specific heat as a specific heat density in 2D. In order to find the specific heat, we should also integrate along the transverse direction (along the circumference of the Bose surface) in momentum space. As the patches are decoupled, the integral is trivial:

$$C_{2D} = \int_{\mathrm{BS}} \frac{dk_\perp}{2\pi} C_{1D} = C_{1D}\frac{2\pi k_B}{2\pi} = \frac{\pi T k_B}{3v}. \tag{35}$$

This should be compared to the specific heat of the Fermi liquid, $C = \frac{\pi^2}{2} n(T/T_F)$ – the results are parametrically equivalent. Note in both expressions I set Boltzmann's constant equal to 1 (not to be confused with the Bose momentum k_B).

Next we consider compressibility. In terms of the density-density correlation function, $\chi_\rho(k,\omega)$, we have that the compressibility κ is given by $\kappa = \chi_\rho(k=0,\omega\to 0)$. Density fluctuations are defined in terms of $\partial_\tau\phi_\gamma$ in $\mathcal{L}_0$ (rather than $\nabla_\gamma\theta_\gamma$, as would be conventional in a Hamiltonian formulation). The leading-order contribution to the density-density correlation function is

$$\chi_\rho(\mathbf{k},\omega) = 2\left(\frac{k_B}{4\pi N\eta v}\right)^2 \sum_{\gamma,\gamma'} \omega^2 \langle \phi_\gamma(\mathbf{k},\omega), \phi_{\gamma'}(-\mathbf{k},\omega)\rangle. \tag{36}$$

Here the Landau parameters will make an $\mathcal{O}(1)$ contribution, so we'll want to keep them in. For the purpose of this density-density correlation function, however, we only have to consider the zeroth Fourier mode of the Landau parameter $f_\rho^{(0)}$. This is because the parameter $f_j(\gamma-\gamma')$ does not contribute to the $\phi\phi$ correlation function at $k=0$ and χ_ρ is

proportional to

$$\frac{1}{N^2}\sum_{\gamma,\gamma'}\langle\phi_\gamma(\mathbf{k},\omega),\phi_{\gamma'}(-\mathbf{k},\omega)\rangle = \langle\phi_0(\mathbf{k},\omega)\phi_0(-\mathbf{k},\omega)\rangle \tag{37}$$

where $\phi_0 = \int \frac{d\gamma}{2\pi}\phi_\gamma$ is the zeroth Fourier mode. This makes physical sense because this is the only conserved density in the problem, as discussed in the stability section. Making use of the correlation function computed in Appendix A,

$$\sum_{\gamma,\gamma'}\langle\phi_\gamma(\mathbf{k},\omega),\phi_{\gamma'}(-\mathbf{k},\omega)\rangle = \\ 2\pi v\eta l_\Lambda \sum_{\gamma,\gamma'}\left(\frac{\delta_{\gamma,\gamma'}}{\omega^2+v^2k_\gamma^2} - \frac{1}{N}\frac{1}{(\omega^2+v^2k_\gamma^2)(\omega^2+v^2k_{\gamma'}^2)}\frac{\omega^2 f_\rho^{(0)}}{1+f_\rho^{(0)}|\omega|/\sqrt{\omega^2+v^2k^2}}\right). \tag{38}$$

Now (after evaluating that Kronecker delta function) we take the sums to integrals, restoring some factors of N:

$$\to 2\pi v\eta l_\Lambda N\int\frac{d\gamma}{2\pi}\left(\frac{1}{\omega^2+v^2k_\gamma^2} - \int\frac{d\gamma'}{2\pi}\frac{1}{(\omega^2+v^2k_\gamma^2)(\omega^2+v^2k_{\gamma'}^2)}\frac{\omega^2 f_\rho^{(0)}}{1+f_\rho^{(0)}|\omega|/\sqrt{\omega^2+v^2k^2}}\right). \tag{39}$$

Given that we're only interested in the $\mathbf{k}=0$ value of the correlator, it will save us a lot of trouble to just set $k_\gamma = 0$ throughout. Both integrals are then trivial, giving us

$$\sum_{\gamma,\gamma'}\langle\phi_\gamma(0,\omega),\phi_{\gamma'}(0,\omega)\rangle = \frac{2\pi v\eta l_\Lambda N}{\omega^2}\left(1-\frac{f_\rho^{(0)}}{1+f_\rho^{(0)}}\right). \tag{40}$$

Collecting terms, we find

$$\chi_{\rho\rho}(0,\omega) = 4\pi v\eta l_\Lambda N\left(\frac{k_B}{4\pi N\eta v}\right)^2\left(1-\frac{f_\rho^{(0)}}{1+f_\rho^{(0)}}\right) = \frac{k_B}{4\pi\eta v}\left(\frac{1}{1+f_\rho^{(0)}}\right), \tag{41}$$

where in the last equality we used the fact that $l_\Lambda = N/k_B$. There is of course no need to take the $\omega\to 0$ limit obtain κ. Compare this result to the Fermi liquid result, $\kappa = \frac{1}{n}\frac{N(0)}{1+F_0^S}$, and we can see that the effect of the Landau parameters is the same.

Now let's work out the dispersion of the zero sound modes. As stated in the Introduction, charge and momentum are carried by separate fields. This means we'll have two collective modes to work out – one for the ϕ_γ fields and one for the θ_γ fields. We'll find that these are related to the two independent Landau parameters. Here we'll make an assumption that the Landau parameters take constant values – in principle it would be possible to get

contributions from higher angular momentum modes if we do not assume this. Then we have the Lagrangian

$$\mathcal{L} = \frac{k_B}{4\pi N\eta}\sum_{\gamma\gamma'}\phi_\gamma\bigg(v^{-1}\omega^2\big(\delta_{\gamma\gamma'}+\frac{1}{N}f_\rho\big) - vk_\gamma k_{\gamma'}\big(\delta_{\gamma\gamma'}+\frac{1}{N}f_j\big)\bigg)\phi_{\gamma'}. \tag{42}$$

Note that, contrary to what we've done up until now, we're working with real frequencies. We can derive the equation of motion for ϕ_γ by requiring that $\delta\mathcal{L}/\delta\phi_\gamma = 0$:

$$0 = (\omega^2 - v^2k_\gamma^2)\phi_\gamma + \frac{1}{N}\sum_{\gamma'}\big(\omega^2 f_\rho - v^2k_\gamma k_{\gamma'}f_j\big)\phi_{\gamma'}. \tag{43}$$

Now we rearrange and sum over γ, explicitly deriving the equation of motion for the zeroth Fourier component, ϕ_0:

$$\phi_0 = -\int\frac{d\gamma}{2\pi}\bigg(\frac{1}{\omega^2 - v^2k_\gamma^2}\bigg)\int\frac{d\gamma'}{2\pi}\big(\omega^2 f_\rho - v^2k_\gamma k_{\gamma'}f_j\big)\phi_{\gamma'}. \tag{44}$$

The f_j term vanishes upon reversing the order of integration and integrating over γ (the integrand is of the form $\cos(\gamma)/(1+\cos^2(\gamma))$). Thus the integral over γ' is trivial, taking $\phi_{\gamma'}\to\phi_0$. The ϕ_0's then drop out, giving us the condition

$$1 = -\omega^2 f_\rho\int\frac{d\gamma}{2\pi}\bigg(\frac{1}{\omega^2 - v^2k_\gamma^2}\bigg) = -\frac{|\omega|f_\rho}{\sqrt{\omega^2 - v^2k^2}}. \tag{45}$$

Thus we find

$$f_\rho = -\sqrt{1-\bigg(\frac{vk}{\omega}\bigg)^2}\in(-1,0) \qquad\Longrightarrow\qquad \omega = \frac{v|k|}{\sqrt{1-f_\rho^2}} \;\;\forall\;\; f_\rho\in(-1,0). \tag{46}$$

The zero sound mode is only defined for $-1 < f_\rho < 0$ and has a dispersion $\omega > v|k|$.

In order to solve for the other zero sound mode, we want to rewrite the Lagrangian in terms of the θ_γ fields. I show how this is done in Appendix C and quote the result,

$$\mathcal{L} = \frac{k_B\eta}{4\pi N}\sum_{\gamma,\gamma'}\theta_\gamma\bigg(v^{-1}\omega^2\big(\delta_{\gamma\gamma'}+\frac{1}{N}\tilde{f}_\rho\big) - vk_\gamma k_{\gamma'}\big(\delta_{\gamma\gamma'}+\frac{1}{N}\tilde{f}_j\big)\bigg)\theta_{\gamma'}, \tag{47}$$

where the dual Landau parameters are given by $f_\rho = -\frac{f_j}{1+f_j}$ and $f_j = -\frac{f_\rho}{1+f_\rho}$. The calculation is equivalent to the one above, so we can see from inspection that the θ_0 zero sound mode exists for $-1 < \tilde{f}_\rho < 0$ (i.e. for $f_j > 0$ in terms of the original parameters) with a dispersion $\omega = v|k|/\sqrt{1-\tilde{f}_\rho^2}$. The general storyline is that couplings between the charge densities, $\partial_\tau\phi$, give rise to collective phase modes (ϕ_0), while couplings between current densities, $\partial_\tau\theta_\gamma$, give rise to collective density modes (θ_0).

We now move to a calculation of the real-space correlation function at large separations. The authors' section on this seems unnecessarily complicated. The basic idea is that, at large spatial distances $x \sim k_B/\Lambda^2$, we need some way of accounting for the curvature of the Bose surface. We will therefore want a decomposition in terms of infinitesimal patches such that we can integrate over γ. We define new patch fields

$$\psi(x,\tau) = \int \frac{d\gamma}{2\pi} e^{ik_B \hat{\gamma}\cdot x} \tilde{\psi}_\gamma(x,\tau) \tag{48}$$

which are similar to the previous patch fields ψ_γ except that they are only supported on an infinitesimal sliver of momentum space in the $\hat{\gamma}_\perp$ direction. Again we'll decompose these into $\tilde{\phi}_\gamma$ and $\tilde{\theta}_\gamma$ fields to represent the low-energy fluctuations. The correlation function between vertex operators $e^{i\tilde{\phi}_\gamma}$ is equivalent to the result of Eq. (18) while taking $\Lambda_\perp \to \infty$, which amounts to taking $\delta_\Lambda(x_\perp) \to 1$. Thus the correlation function is completely independent of the coordinate $x_\perp$. We can then solve for the correlation function between UV bosons:

$$\begin{aligned}
\langle \psi(x,\tau)\psi^\dagger(0)\rangle &= \int \frac{d\gamma}{2\pi} \int \frac{d\gamma'}{2\pi} e^{ik_B\hat{\gamma}\cdot x} \langle \tilde{\psi}_\gamma(x,\tau)\tilde{\psi}^\dagger_{\gamma'}(0)\rangle \\
&\sim \int \frac{d\gamma}{2\pi} \int \frac{d\gamma'}{2\pi} e^{ik_B\hat{\gamma}\cdot x} \langle e^{i\tilde{\phi}_\gamma(x,\tau)} e^{-i\tilde{\phi}_{\gamma'}(0)}\rangle \\
&\sim \int \frac{d\gamma}{2\pi}\ e^{ik_B\hat{\gamma}\cdot x} \frac{1}{(x_\gamma^2 + v^2\tau^2)^{\eta/2}}
\end{aligned} \tag{49}$$

where we have omitted overall factors (and, to be clear, the integral over γ runs from $-\pi$ to π). Now the authors make a useful approximation. The quantity $k_B x \sim (k_B/\Lambda)^2 \gg 1$ so the integrand will oscillate rapidly at points for which $\cos(\gamma)$ is changing as a function of γ (we've taken $x_\gamma = x\cos(\gamma)$). We can therefore restrict our attention to the points where $\cos(\gamma)$ is stationary, $\gamma = 0, \pm\pi$. At these points $\cos(\gamma) \approx 1$, so we can make this approximation in the denominator and take it to be independent of γ. We therefore obtain, for large separations,

$$\langle \psi(x,\tau)\psi^\dagger(0)\rangle \sim \frac{J_0(k_B x)}{(x_\gamma^2 + v^2\tau^2)^{\eta/2}} \tag{50}$$

where $J_0(x)$ is a Bessel function of the first kind. Setting $\tau = 0$, we expand the Bessel function for large arguments to get the asymptotic form

$$\langle \psi(x,\tau)\psi^\dagger(0)\rangle \sim \frac{\cos(k_B x - \pi/4)}{x^{\eta+1/2}}. \tag{51}$$

If one were to take $\eta = 1$ and $k_B \to k_F$, this would be same asymptotic form observed in Fermi liquids: the correlation functions oscillate with a frequency set by the Bose momentum,

offset by a $\pi/4$ phase shift. The UV bosons have a correlation function that decays more quickly with distance than the patch field bosons (which have correlations that fall off as $\sim x^{-\eta}$) due to destructive interference from different parts of the Bose surface.

Finally, to further characterize this phase of matter, we will determine whether it is a metal, a superfluid, or an insulator. We follow the procedure set out in Ref. [14]: minimally coupling to a gauge field and then taking static limits. The formal procedure will be to integrate out the ϕ fields and obtain an effective action for the gauge field, A, before taking the limits. The coupled Lagrangian is

$$\mathcal{L}[A] = \frac{k_B}{4\pi N\eta}\sum_\gamma \big(v^{-1}\big(\partial_\tau\phi_\gamma - A_\tau\big)^2 + v\big(\nabla_\gamma\phi_\gamma - \mathbf{A}\cdot\hat{\gamma}\big)^2\big), \tag{52}$$

where the authors show that this is equivalent to coupling the field A directly to the zeroth Fourier mode, ϕ_0. Again this makes sense because this is the only Fourier mode charged under the microscopic U(1) symmetry. We immediately proceed to integrate out the ϕ_γ fields. We'll do this explicitly, defining $S = \frac{1}{N}\sum_\gamma S_\gamma$:

$$\begin{aligned} S_\gamma &= \frac{k_B}{4\pi\eta}\int d^2x\ d\tau\ \big(v^{-1}\big(\partial_\tau\phi_\gamma - A_\tau\big)^2 + v\big(\nabla_\gamma\phi_\gamma - \mathbf{A}\cdot\hat{\gamma}\big)^2\big) \\ &= \frac{k_B}{4\pi\eta}\int \frac{d^2k\ d\omega}{(2\pi)^3}\ \big(v^{-1}\big(\omega\phi_\gamma - A_\tau\big)^2 + v\big(k_\gamma\phi_\gamma - \mathbf{A}\cdot\hat{\gamma}\big)^2\big) \\ &= \frac{k_B}{4\pi\eta}\int \frac{d^2k\ d\omega}{(2\pi)^3}\ \big((v^{-1}\omega^2 + vk_\gamma^2)\phi_\gamma^2 - 2(v^{-1}\omega A_\tau + vk_\gamma A_\gamma)\phi_\gamma + v^{-1}A_\tau^2 + vA_\gamma^2\big). \end{aligned} \tag{53}$$

We now complete the square by adding and subtracting a term quadratic in A, such that one could perform a trivial Gaussian integral over the ϕ_γ fields in the path integral. Removing this purely quadratic term, we obtain the effective Lagrangian for the A field:

$$S_\gamma = \frac{k_B}{4\pi\eta}\int \frac{d^2k\ d\omega}{(2\pi)^3}\ \left(v^{-1}A_\tau^2 + vA_\gamma^2 - \frac{v^{-1}}{\omega^2 + v^2k_\gamma^2}(A_\gamma k_\gamma v^2 + A_\tau\omega)^2\right). \tag{54}$$

Cleaning this up a bit, we can pull out an effective Lagrangian in terms of A:

$$\mathcal{L}_{\text{eff}}[A] = \frac{k_B v}{4\pi\eta}\int \frac{d\gamma}{2\pi}\frac{A_\gamma^2\omega^2 + A_\tau^2k_\gamma^2 - 2A_\tau A_\gamma k_\gamma\omega}{\omega^2 + v^2k_\gamma^2}. \tag{55}$$

Note that we took $N\to\infty$ above such that we integrate over γ. We'll work in the Coulomb gauge, imposing $\mathbf{k}\cdot\mathbf{A} = 0$. Thus, if we identify $k_\gamma = k\cos(\gamma)$, then we should take $A_\gamma = A\sin(\gamma)$. The Lagrangian therefore takes the form

$$\begin{aligned} \mathcal{L}_{\text{eff}}[A] &= \frac{k_B v}{4\pi\eta}\int \frac{d\gamma}{2\pi}\frac{(A\omega\sin(\gamma) - A_\tau k\cos(\gamma))^2}{\omega^2 + v^2k^2\cos^2(\gamma)} \\ &= \frac{k_B v}{4\pi\eta}\left(\frac{A_\tau^2}{v^2}\left(1 - \frac{1}{\sqrt{1+\xi^2}}\right) + \frac{A^2}{\xi^2}\left(\sqrt{1+\xi^2} - 1\right)\right) \end{aligned} \tag{56}$$

where $\xi = vk/\omega$.

Given this expression we will set $A_\tau = 0$ and take limits of $\omega \to 0$ and $k \to 0$ in both orders. The limit in which $\omega = 0$ first corresponds to transverse response (analogously, think of $B = \nabla \times A$ remaining finite while $E = \partial A/\partial t$ goes to zero), and therefore a finite value to this limit will correspond to a finite superfluid density. This is analogous to the Meissner effect. Specifically, the effective Lagrangian in this limit will be equal to $\frac{1}{2\pi} D_s A^2$ where superfluid weight $D_s = \pi n_s/m$ [14] and n_s is the superfluid density. In this expression above, this limit corresponds to $\xi \to \infty$, in which case $\mathcal{L}_{\text{eff}}[A] \to 0$: the Bose-Luttinger Liquid is not a superfluid.

The opposite limit, where $k = 0$ first, can be thought of as the response to a DC electric field. The result in this limit is equivalent to the above result except with D_s replaced by $D = \pi n/m$, the Drude weight (n is the density of charge carriers). The Drude weight is the prefactor to the delta function contribution to the low-temperature conductivity, $\sigma(\omega) = D\delta(\omega) + \sigma_{reg}(\omega)$. This delta function peak at zero frequency is a characteristic of (clean) metals at zero temperature, when the scattering lifetime diverges. This limit corresponds to $\xi \to 0$ and $\mathcal{L}_{\text{eff}}[A] \to \frac{k_B v A^2}{8\pi\eta}$: the Bose-Luttinger liquid is a metal! From this we can extract the Drude weight of the Bose-Luttinger liquid, $D = k_B v/4\eta$. One might look at this result and attempt to pull out an effective mass $m = k_B/v$ and effective carrier density $n = k_B^2/4\pi\eta$ (as the authors do), but seeing as these quantities don't show up anywhere else I'll let the reader draw their own conclusions.

V. FINITE DENSITY

Up until now, we have been discussing a particle-hole symmetric UV theory with zero background density. If our result that the BLL is a compressible, metallic phase of matter is to be sensible, we should determine that these results hold at fixed density.

Particle-hole symmetry enforced that the Lagrangian could only contain a ∂_τ^2 term – no single time derivatives, ∂_τ. A Lagrangian at fixed density would instead take the form

$$\mathcal{L} = \psi^* \bigg(-(\partial_\tau + \mu) + \frac{1}{8mk_B^2}\big(-\nabla^2 - k_B^2\big)^2 \bigg) + \frac{g}{4}|\psi|^4. \tag{57}$$

Note that the squared time derivative has been removed here. This is because, with a single time derivative, we have to rescale space and time by different factors of Λ'/Λ under the

renormalization group. Essentially we have to fix the rules of the game so that the free Lagrangian is invariant under rescaling. If we had kept both the ∂_τ^2 and ∂_τ terms in and rescaled ω and k by one power of Λ'/Λ, then we would find that the ∂_τ term grows faster than the ∇^2 term. This would mean that our kinetic term is irrelevant, which we don't want. We therefore rescale ω by $(\Lambda/\Lambda')^2$ (which is to say that we use a dynamical exponent $z = 2$) in order to ensure that the kinetic term is marginal. Under this rescaling, we would then find that the ∂_τ^2 term is irrelevant – and thus, it is not included in Eq. (57).

The analysis of this Lagrangian is very similar. We can clearly expand in terms of different patch fields, ψ_γ, as before. We assume $\mu > 0$ (equivalent to $r < 0$ in Eq. (8)), so we can then expand each ψ_γ in terms of amplitude and phase fluctuations. These can be rewritten in terms of the conjugate variables θ_γ and ϕ_γ,

$$\psi_\gamma(x) = \sqrt{\rho_0 + \frac{k_B}{2\pi}\nabla_\gamma\theta_\gamma}\; e^{i\phi_\gamma}. \tag{58}$$

We might now worry about how θ transforms under translation, $\theta(x) \to \theta(x+\mu) + \frac{\rho_0}{k_B}\hat{\gamma}\cdot\mu$. This is because $\theta(x)$ is the integrated density (it's gradient corresponds to density fluctuations), so with a finite average density it must change as a function of position. The fixed point Lagrangian and the most relevant perturbations are not altered by this, however, so this doesn't change any of our analysis. Expanding Eq. (57) in these variables leads to a Lagrangian of the form

$$\begin{aligned}\tilde{\mathcal{L}}_0 = \frac{1}{N}\sum_\gamma \left(\frac{k_B}{2\pi}\nabla_\gamma\theta_\gamma\partial_\tau\phi_\gamma + \beta(\nabla_\gamma\phi_\gamma)^2\right) \\ + \frac{1}{N^2}\sum_{\gamma,\gamma'}\left(g_\theta(\gamma-\gamma')(\nabla_\gamma\theta_\gamma)(\nabla_{\gamma'}\theta_{\gamma'}) + g_\phi(\gamma-\gamma')(\nabla_\gamma\phi_\gamma)(\nabla_{\gamma'}\phi_{\gamma'})\right)\end{aligned} \tag{59}$$

We get no $(\partial_\tau\phi_\gamma)^2$ term because there is no ∂_τ^2 term in the UV Lagrangian. Instead, the ∂_τ term gives us a coupling between $\nabla_\gamma\theta_\gamma$ and $\partial_\tau\phi_\gamma$. If we were now to integrate out the θ_γ fields, we would find that the $(\nabla_\gamma\theta_\gamma)(\partial_\tau\phi_\gamma)$ term would simply rescale the term coupling $\partial_\tau\phi_\gamma$ on different patches (i.e. it would just change the definition of the vector b discussed in Appendix C). In the absence of a $(\partial_\tau\phi_\gamma)^2$ term, then, the form of the effective Lagrangian for the ϕ fields would be

$$\tilde{\mathcal{L}}_0[\phi] = \frac{k_B}{4\pi\eta N}\sum_\gamma v(\nabla_\gamma\phi_\gamma)^2 + \frac{k_B}{4\pi\eta N^2}\sum_{\gamma,\gamma'}\left(v^{-1}f_\rho^{\gamma,\gamma'}(\partial_\tau\phi_\gamma)(\partial_\tau\phi_{\gamma'}) + v f_j^{\gamma,\gamma'}(\nabla_\gamma\phi_\gamma)(\nabla_{\gamma'}\phi_{\gamma'})\right). \tag{60}$$

At this point the authors make a physical argument. Based on the results we have already derived, we know that the $(\partial_\tau \phi)^2$ terms lead to charge stiffness in the Lagrangian. Furthermore, it was already shown that the Landau parameters only make an $\mathcal{O}(1/N)$ contribution to the correlation function. Thus, the Lagrangian above has no terms making an $\mathcal{O}(N^0)$ contribution to the charge stiffness. This should trouble us because N flows under RG, so it is as unphysical as the cutoff Λ. In particular, as the correlation function will be N-dependent to leading order, we will derive scaling exponents that are N-dependent. We should not find that physical observables depend on N, just as we should not find that they depend on Λ. On the basis of this observation, the authors deem it imperative to keep the ∂_τ^2 term in the UV Lagrangian, even if it is irrelevant.

This may seem a bit far-fetched, but let's see how it plays out. Consider a new UV Lagrangian given by

$$\mathcal{L} = \psi^* \bigg(-(\partial_\tau + \mu) - \frac{\lambda m^2}{k_B^2}\partial_\tau^2 + \frac{1}{8mk_B^2}\big(-\nabla^2 - k_B^2\big)^2 \bigg) + \frac{g}{4}|\psi|^4. \tag{61}$$

As stated from the outset, under the $z = 2$ RG scaling the parameter λ is irrelevant. Keeping it in, however, it contributes a $(\partial_\tau \phi_\gamma)^2$ term to the IR Lagrangian. Notice, now, that the single time derivative term renormalized the charge density coupling term $\sim (\partial_\tau \phi_\gamma)(\partial_\tau \phi_{\gamma'})$ after we integrated out the θ_γ fields. *This term rescales in the same way as the* $(\partial_\tau \phi_\gamma)^2$ *under RG with any dynamical exponent.* This means that, although λ was an irrelevant coupling the UV Lagrangian with $z = 2$ scaling, we find that it has precisely the same scaling dimension as the ∂_τ in the IR Lagrangian. For that reason, we should keep the ∂_τ^2 term in the UV Lagrangian with finite density, and the IR Lagrangian assumes a form identical to Eqs. (8) and (9). Our prior analysis of the fixed point therefore applies just as well at finite density.

VI. $3+1$D FIXED POINT

At this point we have finished the analysis of the $2+1$D model. This section is meant to assure the reader that the exact same analysis can be carried out in $3+1$D. We consider the same UV Lagrangian in Eq. (1). The patch decomposition goes through in an identical manner, where the patches are now cubes of volume Λ^3 and are labeled by a unit vector $\hat{\gamma}$. There are a total of $N = 4\pi k_B^2/\Lambda^2$ patches. We again break up the UV boson fields $\psi(x)$

into patch fields. Expanding the kinetic energy term again yields a quasi-1D dispersion, which takes us to the same patch Lagrangian given by Eq. (4). The kinetic terms we discard will differ in 3D, but those should not affect our analysis to leading order. There are also additional scattering processes that are allowed in 3D (non-forward scattering), but these will always be less relevant than the BCS coupling [1].

Now we make the same amplitude-phase decomposition and rewrite the Lagrangian in terms of ϕ_γ and θ_γ fields. As our patch Lagrangian is of the same form, it should be no surprise that the fixed point Lagrangian is also identical. The derivation of the correlation functions from Sec. III follows through in exactly the same manner. The only difference is that we have two directions perpendicular to $\hat{\gamma}$, so we instead find

$$\langle e^{i\phi_\gamma(x)} e^{-i\phi_{\gamma'}(0)} \rangle \sim \frac{\delta_{\gamma,\gamma'} \delta_\Lambda(x_{\perp,1}) \delta_\Lambda(x_{\perp,2})}{((\Lambda \hat{\gamma} \cdot \vec{x})^2 + (1 + \Lambda v \tau)^2)^{\eta/2}} \tag{62}$$

with $\delta_\Lambda(x)$ functions constraining both transverse displacements (by my definitions $x_{\perp,1}$, $x_{\perp,2}$ and $\hat{\gamma} \cdot \vec{x}$ are the lengths of the three orthogonal components of $\vec{x}$).

Turning to the question of stability, the most relevant perturbation is again the BCS pairing term:

$$\frac{1}{N^2} \sum_{\gamma,\gamma'} g_{\text{BCS}}(\hat{\gamma} \cdot \hat{\gamma}') \cos\big(\phi_\gamma(r) + \phi_{\gamma+\pi}(r) - \phi_{\gamma'}(r) - \phi_{\gamma'+\pi}(r)\big), \tag{63}$$

where it should be understood that γ and $\gamma+\pi$ indicate antipodal points on the Bose surface. The first point to make is a rehashing of a point emphasized in Sec. III: even though there are four spacetime dimensions in this problem, the rescaling of two of the momentum directions ($\hat{k}_{\perp,1}$ and $\hat{k}_{\perp,2}$) will be cancelled out by a proportionate increase in the number of patches, N. This means that the scaling dimension of the operator $\cos(\varphi_{\gamma,\gamma'})$ in the 3D case should still be compared to 2 rather than 4, as was the case in the $2+1$D problem. With that remark, the tree-level stability analysis is identical to the $2+1$D problem (as we are only eliminating modes from the effective $1+1$D problem) and gives an RG eigenvalue of $2-2\eta$. Thus, the $3+1$D BLL fixed point is stable for $\eta > 1$, just as in 2D.

So we find that pretty much everything is the same. We shouldn't expect that this is the case for everything, however. In particular, one would expect there to be a few phenomenological differences between 2D and 3D. The one the authors mention is the UV boson correlation function, which should have a slightly different asymptotic form. We can again imagine breaking up the Bose surface into infinitesimal chunks that we'll integrate

over. We already solved for the correlation function, and we know the correlation function of the $\tilde{\phi}_\gamma$ variables living on the infinitesimal slivers will be identical except that they will be completely independent of the transverse directions $x_{\perp,1}$ and $x_{\perp,2}$. We therefore find

$$\begin{aligned}\langle\psi(x,\tau)\psi^\dagger(0)\rangle &\sim \int \frac{d\gamma}{4\pi}\int \frac{d\gamma'}{4\pi} e^{ik_B\hat{\gamma}\cdot x}\langle e^{i\tilde{\phi}_\gamma(x,\tau)}e^{-i\tilde{\phi}_{\gamma'}(0)}\rangle \\ &\sim \int \frac{d\gamma}{4\pi} e^{ik_B\hat{\gamma}\cdot x}\frac{1}{(x_\gamma^2+v^2\tau^2)^{\eta/2}}\end{aligned} \tag{64}$$

Now note that $d\gamma$ is a solid angle over the Bose sphere. Picking the polar axis along $\hat{x}$ and defining polar and azimuthal angles θ and ϕ, respectively (apologies for the overlap with field variable names), we can rewrite this as

$$\begin{aligned}\langle\psi(x,\tau)\psi^\dagger(0)\rangle &\sim \int d\phi\; d(\cos(\theta))\; e^{ik_B x\cos(\theta)}\frac{1}{(x^2\cos^2(\theta)+v^2\tau^2)^{\eta/2}} \\ &\sim \int_{-1}^{1} dz\; \frac{e^{ik_B xz}}{(x^2z^2+v^2\tau^2)^{\eta/2}} \\ &\sim \frac{\sin(k_B x)}{x^{\eta+1}}\end{aligned} \tag{65}$$

We took $\tau = 0$ in the last step. We still see spatial oscillations at k_B, but the phase has shifted by $\pi/4$ from the 2D case. We also get more destructive interference by integrating patch contributions over a sphere rather than a circle, so the correlator falls off more quickly than in 2D.

VII. EXPERIMENTAL REALIZATION

This paper proposed the existence of a novel form of matter, so an important question to address is where we might look to find it. I will briefly address some possibilities, although there are far more questions than answers in this section.

A. Metallic Helimagnet

These are ferromagnetic materials such as manganese monosilicide (MnSi) and iron germanide (FeGe) whose spins align in a helical pattern at low temperatures. The origin of this spiral pattern is the Dzyaloshinskii–Moriya (DM) interaction, which is an interaction between spins of the form $\int \vec{S}\cdot(\nabla\times\vec{S})$. By applying pressure, it has been observed that MnSi exhibits a phase transition from the ferromagnetic phase with this helical spin ordering

to a paramagnetic phase with partial long-range magnetic ordering [15]. More specifically, the DM interaction induces spiral ordering of spins which, due to other spin-orbit interactions, is locked along the reciprocal-space direction $\mathbf{Q} = \langle 111 \rangle$ in the ferromagnetic phase. In Ref. [15] this is determined by neutron diffraction measurements. Upon crossing the ferromagnetic transition, one would assume that the magnitude of the magnetic moment (and hence the spiral ordering as well) should vanish. Instead, the authors measure scattering intensity across a small sphere in momentum space. Rather than localized Bragg-like peaks, they find that the scattering intensity is (not quite uniformly) smeared out. They interpret this as the same spiral order parameter whose magnitude survives above the ferromagnetic transition, but whose direction is no longer pinned in reciprocal space. The state is therefore paramagnetic (insofar as it is decidedly not ferromagnetic) but possesses some kind of pseudo-long-range order. Additionally, in this region the material displays non-Fermi-liquid behavior with a resistivity that vanishes as $T^{3/2}$ [16].

As this result is indicative of a bosonic excitation living on the surface of a sphere in momentum space, the authors of Ref. [5] claim that it could be described by the $3+1$D BLL fixed point. They then proceed to show that coupling the BLL to a free electron gas would lead to a resistivity proportional to T^{η}, indicating that a BLL with $\eta = 3/2$ (which is within the stable $\eta > 1$ regime) could describe the observed results. While I have not found evidence that this phenomenon has been widely observed in other metallic helimagnets, it raises the possibility of a condensed matter environment that may host BLL-type physics.

B. Spin-Orbit Coupled Bosons

A proposal that the authors only cursively address here but has been mentioned elsewhere [17] is the use of spin-orbit coupled bosons to realize this dispersion relation. Such systems have already been engineered in cold atoms through Raman dressing of the hyperfine atomic energy levels (these hyperfine levels can be considered pseudospin states) [18–22]. It has been determined that the lower band of Rashba spin-orbit coupled bosons (whose coupling is of the form $\sim \vec{p} \cdot \vec{\sigma}$) would have minima along a circle in $2+1$D [23] and a sphere in $3+1$D [24]. This system would therefore appear to realize the UV Lagrangian in Eq. (4). In general it is non-trivial to engineer this Rashba coupling because the Raman-dressing scheme more naturally realizes some combination of Rashba and Dresselhaus couplings. This would

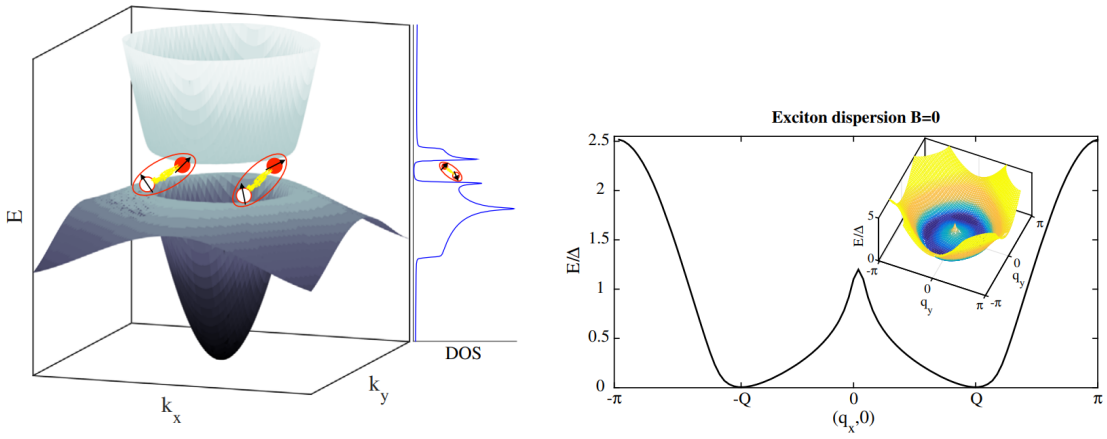


FIG. 2. Plot of the localized and itinerant bands (first panel) and the exciton dispersion relation resulting from an onsite interaction term. Taken from Ref. [25].

mean that the dispersion relation tends to be anisotropic, having a single minimum in momentum space (e.g. Fig. 2 of Ref. [18]). With that said, nothing precludes the generation of a nearly-isotropic dispersion relation. Although the authors of Ref. [5] claim that the BLL would be stable to weak perturbations of the circular Bose surface, it is unclear how it would fare if the *dispersion minima* were nearly but not completely degenerate. Such a system would otherwise be an ideal candidate for realizing the BLL, as cold atom systems could realize the low-density, weakly-interacting limit about which we expanded in this paper.

C. Excitons in Spin-Orbit Coupled Fermi System

One might also imagine that excitons, bosonic bound states of particles and holes, could realize the BLL fixed point. To that end, Ref. [25] studies exciton formation in so-called Kondo insulators. Kondo insulators are materials in which itinerant electrons hybridize with localized spins to form a heavy band insulator. The specific material they consider is samarium hexaboride, SmB_6. They model the system by considering itinerant d electrons that hybridize with localized f electrons due to stong spin-orbit coupling. The d and f bands are shown in the first panel of Fig. 2. Note that they already possess a ring-like dispersion due to the spin-orbit coupling. One can imagine that excitons would form along the ring where the bands are closest. Furthermore, they show that the density of states is peaked near these band edges – a further indication that the system would be susceptible to exciton formation.

The authors then consider a Hubbard-like interaction between f electrons, which can lead to the formation of excitons. Considering just the terms contributing to exciton formation, they find the exciton dispersion shown in the second panel of Fig. 2. This dispersion has, again, a *nearly* degenerate ring of minima at fixed quasimomentum. The authors find that the lattice lifts the degeneracy, so in this particular setup one would again have to reckon with the question raised in the section above.

There are a few other matters that further complicate the exciton proposal. One question would be the nature of interactions between excitons, which have proven challenging to understand [26]. Our approach to the BLL involved weak, short-ranged UV interactions. This seems like a reasonable approximation for neutral particle-hole pairs, but without a proper accounting it is unclear what complications might arise. Another would be the role of a finite lifetime: while the authors demonstrate that the BLL fixed point does not require U(1) charge conservation, it would not be outlandish to imagine that decay and recombination processes might change our analysis.

D. Bosons on a Honeycomb Lattice

A particularly clean example of a dispersion with such degenerate minima is a honeycomb lattice with nearest-neighbor and next-nearest-neighbor hopping. Others have studied hard core bosons on this lattice [27–29] which maps onto a frustrated XY model. It has been argued that this model realizes a Bose metal phase where the momentum occupation displays a peak in a circular Bose surface. The authors of Ref. [5] attempt to distinguish the BLL from a Bose metal insofar as the BLL is a more generally formulated on the notion of a Bose surface hosting collective gapless modes. With that said, the realization in Sec. IV that the BLL is metallic, and thus an example of a Bose metal, muddies the distinction.

Solving for the bands is simple, but if one doesn't want to bother with that there's a clean form written out in Ref. [27]. Three examples are shown in Fig. 3 for $t_2/t_1 = 0.18$, 0.25 and 0.4. The honeycomb lattice with next-nearest-neighbor hopping has minima along a distorted ring centered about $\vec{k} = 0$ for $1/6 \leq t_2/t_1 \leq 1/2$. As t_2/t_1 approaches $1/2$, however, the ring begins to develop sharp corners that would be incompatible with our analysis. Conversely, the ring better approximates a circle closer to $t_2/t_1 = 1/6$ but the effective harmonic potential in the radial direction decreases in magnitude. Decreasing this

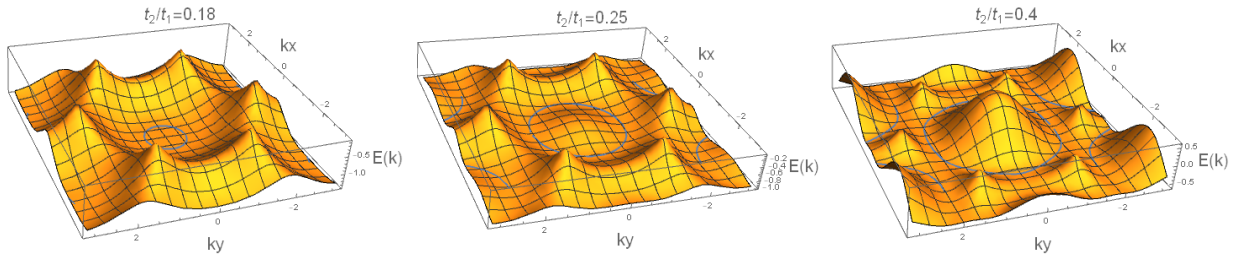


FIG. 3. Plots of the lowest band on a honeycomb lattice with nearest-neighbor hopping t_1 and next-nearest-neighbor hopping t_2. Shown are plots for $t_2/t_1 = 0.18$, 0.25, and 0.4. A blue line indicates the position of the degenerate minima.

harmonic trapping would effectively mean that one would have to work at smaller densities and interaction strengths in order to be justified in flowing to the IR fixed point we identified. With that said, we did not derive bounds on the allowed densities and interaction strengths for which this analysis is valid. Furthermore, results indicating the presence of a Bose surface in strongly-interacting systems should bolster one's confidence in this platform's ability to host a BLL. One should also consider this model compared to the others presented up until now. This appears to be the only model presented where the distortions to the UV Bose surface are in the radial direction, yet the ring of minima are precisely degenerate. It seems likely that radial distortions would be a less severe approximation of the dispersion envisioned here. Optical honeycomb lattices have been realized before [30, 31], although I was not able to find a reference that implemented next-nearest-neighbor hopping.

VIII. CONCLUSIONS

In this paper we have demonstrated the stability of a novel phase of matter. The Bose-Luttinger liquid is a state with gapless excitations along a surface in momentum space, analogous to a Fermi liquid. The similarity extends to many of its phenomenological properties: the BLL is a compressible, metallic phase with a T-linear specific heat; it has Landau parameters coupling different charge and momentum densities on the Bose surface; and the UV boson correlation function exhibits spatial oscillations at a period set by the Bose momentum. As the Bose surface itself does not arise due to degeneracy pressure, however, there is no clear analog of Luttinger's theorem. This means that charge and momentum densities in the BLL are in principle decoupled, which implies the existence of two zero

sound modes instead of one as well as a continuously-tunable parameter η that controls the decay of correlation functions as well as a variety of other properties. More broadly, this means that excitations of the BLL fixed point are not quasiparticles. They should instead be understood as collective density modes, as one would find in a Luttinger liquid.

It is not obvious where one would look to realize this phase of matter. The precisely degenerate and circular dispersion minimum posited in this paper arises in the lower band of Rashba spin-orbit coupled bosons, but Raman-dressing procedures generally lead to Dresselhaus-type contributions as well that would complicate the matter substantially. The authors claim that a BLL fixed point might describe exotic partial long-range magnetic order in the paramagnetic phase of metallic helimagnets, leading to non-Fermi-liquid behavior in the resistivity. While their model can arrive at a resistivity with the same low-temperature scaling, there is a substantial amount of uncertainty on this point. In general it would appear that cold atom systems, with the ability to control interactions and engineer superlattices and artificial spin-orbit coupling, are the best option for realizing this phase.

Appendix A: Landau Parameter Correction to Correlation Function

As mentioned above, in the absence of $\mathcal{L}_f$ the Lagrangian $\mathcal{L}_0$ is diagonal in the ϕ_γ fields and we have

$$G_\phi^{\gamma\gamma'}(\mathbf{k},\omega) = \langle \phi_\gamma(\mathbf{k},\omega)\phi_{\gamma'}(-\mathbf{k},\omega)\rangle = \delta_{\gamma\gamma'}\frac{2\pi v\eta l_\Lambda}{\omega^2 + v^2k_\gamma^2} \tag{A1}$$

where $k_\gamma = \hat{\gamma}\cdot\mathbf{k}$. Note that the authors define the perpendicular length scale of the patch fields $l_\Lambda^{-1} = \int_{-\Lambda}^{\Lambda}\frac{dk_\perp}{2\pi} = \Lambda/\pi = k_B/N$ for convenience. Now let's see how the Landau parameters affect this result. This is in general a considerably more complicated calculation, so we'll just take $f_\rho^{\gamma,\gamma'} = f_\rho$ to be a constant and $f_j^{\gamma,\gamma'} = 0$. We write the Lagrangian out in Fourier space,

$$\mathcal{L} = \frac{k_B}{4\pi N\eta}\sum_{\gamma,\gamma'}\phi_\gamma\big((v^{-1}\omega^2 + vk_\gamma^2)\delta_{\gamma,\gamma'} + \frac{1}{N}v^{-1}\omega^2 f_\rho\big)\phi_{\gamma'}. \tag{A2}$$

This equation is of the form $\phi_\gamma M_{\gamma,\gamma'}\phi_\gamma$. We must invert the matrix M to obtain the propagator, so let's break it down into two parts: $M = B + aC$ where C is an $N\times N$ matrix where every value is $1/N$ and B is diagonal. The strategy here will be to perform a power

expansion and sum the terms to all orders:

$$(B + aC)^{-1} = B^{-1} \sum_{k=0}^{\infty} \big(- aCB^{-1} \big)^k. \tag{A3}$$

Clearly to zeroth order we recover the result above:

$$[B^{-1}]^{(0)}_{\gamma\gamma'} = \delta_{\gamma\gamma'} \frac{2\pi v\eta l_\Lambda}{\omega^2 + v^2 k_\gamma^2} \equiv 2\pi v\eta l_\Lambda G^0_\gamma \delta_{\gamma\gamma'}. \tag{A4}$$

It's not hard to see that, to second order, we simply have the product of the two G^0_γ terms:

$$[B^{-1}]^{(0)}_{\gamma\gamma'} + [B^{-1}]^{(1)}_{\gamma\gamma'} = 2\pi v\eta l_\Lambda \big(G^0_\gamma \delta_{\gamma\gamma'} - \frac{\omega^2 f_\rho}{N} G^0_\gamma G^0_{\gamma'} \big). \tag{A5}$$

Given that fact, the crucial observation is that each successive multiplication by $(-aCB^{-1})$ multiplies the previous term by $\big(-\frac{1}{N}\omega^2 f_\rho \sum_{\gamma''} G^0_{\gamma''} \big)$. This becomes quite clear once realizing that aCB^{-1} is composed of identical rows of the form $[aCB^{-1}]_{\gamma\gamma'} = \frac{1}{N}\omega^2 f_\rho G^0_{\gamma'}$. This gives us the following expression:

$$G^{\gamma\gamma'}_\phi(\mathbf{k}, \omega) = \sum_{l=0}^{\infty} [B^{-1}]^{(l)}_{\gamma\gamma'} = 2\pi v\eta l_\Lambda \bigg(G^0_\gamma \delta_{\gamma,\gamma'} - \frac{\omega^2 f_\rho}{N} G^0_\gamma G^0_{\gamma'} \bigg(\sum_{l=0}^{\infty} \big(- \frac{1}{N}\omega^2 f_\rho \sum_{\gamma''} G^0_{\gamma''} \big)^l \bigg) \bigg). \tag{A6}$$

At this point things look like a mess. Our next task is to perform the sum over γ'', absorbing the factor of N to turn it into an integral:

$$\frac{1}{N} \sum_{\gamma''} G^0_{\gamma''} \to \int \frac{d\gamma}{2\pi} \frac{1}{\omega^2 + v^2 k^2 \cos^2(\gamma)} = \frac{1}{|\omega| \sqrt{\omega^2 + v^2 k^2}}. \tag{A7}$$

Now we can perform the sum over l in isolation,

$$\sum_{l=0}^{\infty} \bigg(- \frac{|\omega| f_\rho}{\sqrt{\omega^2 + v^2 k^2}} \bigg)^l = \frac{1}{1 + f_\rho |\omega| / \sqrt{\omega^2 + v^2 k^2}}, \tag{A8}$$

and by inserting that back into the original formula we arrive at

$$\begin{aligned} G^{\gamma\gamma'}_\phi(\mathbf{k}, \omega) &= 2\pi v\eta l_\Lambda \bigg(G^0_\gamma \delta_{\gamma\gamma'} - \frac{\omega^2 f_\rho}{N} G^0_\gamma G^0_{\gamma'} \frac{1}{1 + f_\rho |\omega| / \sqrt{\omega^2 + v^2 k^2}} \bigg) \\ &= 2\pi v\eta l_\Lambda \bigg(\frac{\delta_{\gamma,\gamma'}}{\omega^2 + v^2 k_\gamma^2} - \frac{1}{N} \frac{1}{(\omega^2 + v^2 k_\gamma^2)(\omega^2 + v^2 k_{\gamma'}^2)} \frac{\omega^2 f_\rho}{1 + f_\rho |\omega| / \sqrt{\omega^2 + v^2 k^2}} \bigg). \end{aligned} \tag{A9}$$

Thus, like the Fermi liquid, the Landau parameters affect the correlation function only at order $1/N$. Seeing as N will grow as we reduce the cutoff, we are prompted to set the

Landau parameters equal to zero for the majority of our calculations. The exception will be in the discussion of certain phenomenological properties for which they make an $\mathcal{O}(1)$ contribution. It's also worth mentioning that this is also a feature of Fermi liquids. That Landau parameters only contribute to the correlation functions (and, importantly, to the self energy) at order $1/N$ is related to the fact that Landau parameters cannot destabilize the Fermi liquid unless they are "sufficiently singular".

Appendix B: One-Loop RG

We want to calculate the correction at one loop, which means calculating the quantity

$$\frac{1}{2}\big(\langle \delta S[\phi^< + \phi^>]^2\rangle_> - \langle \delta S[\phi^< + \phi^>]\rangle_>^2\big) \tag{B1}$$

which appears in the cumulant expansion in Eq. (22). The full term looks like

$$I^{(2)} = \frac{1}{N^4}\sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} g_{\rm BCS}(\gamma_1-\gamma_2)g_{\rm BCS}(\gamma_3-\gamma_4)\int d^3r_1\int d^3r_2\,\times \\ \bigg(\langle\cos(\varphi_{\gamma_1,\gamma_2}(r_1))\cos(\varphi_{\gamma_3,\gamma_4}(r_2))\rangle - \langle\cos(\varphi_{\gamma_1,\gamma_2}(r_1))\rangle\langle\cos(\varphi_{\gamma_3,\gamma_4}(r_2))\rangle\bigg). \tag{B2}$$

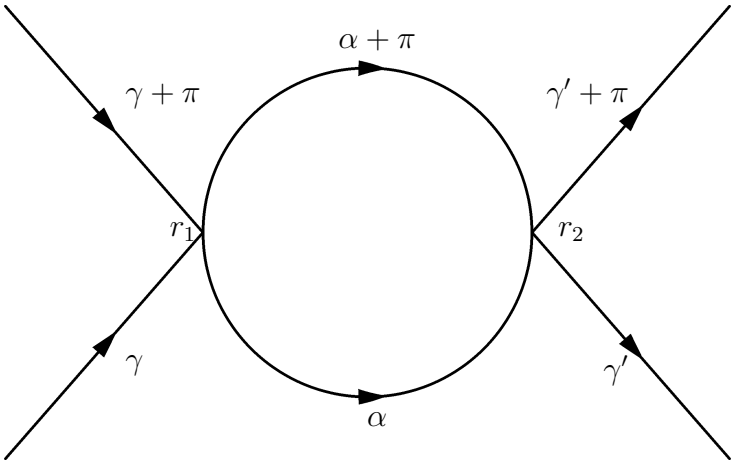
Separating the cosines into complex exponentials, we can rewrite the term in parentheses as

$$\frac{1}{4}\sum_{\epsilon,\epsilon'=\pm}\bigg(\langle e^{i\epsilon\varphi_{\gamma_1,\gamma_2}(r_1)}e^{i\epsilon'\varphi_{\gamma_3,\gamma_4}(r_2)}\rangle - \langle e^{i\epsilon\varphi_{\gamma_1,\gamma_2}(r_1)}\rangle\langle e^{i\epsilon'\varphi_{\gamma_3,\gamma_4}(r_2)}\rangle\bigg) \tag{B3}$$

where one should recall the definition

$$\varphi_{\gamma_1,\gamma_2}(r) = \phi_{\gamma_1}(r) + \phi_{\gamma_1+\pi}(r) - \phi_{\gamma_2}(r) - \phi_{\gamma_2+\pi}(r). \tag{B4}$$

The key thing to note here is that if none of the γ_i are equal then the first term is cancelled by the second term. This would correspond to a disconnected diagram. Similarly if $\gamma_1 = \gamma_2$ or $\gamma_3 = \gamma_4$ but no other terms are equal, the term in parentheses evaluates to zero (it's also true that $\varphi_{\gamma_1,\gamma_2}(r) = 0$ when $\gamma_1 = \gamma_2$). The terms that will renormalize the interaction term will therefore have either γ_1 or γ_2 equal to either γ_3 or γ_4. There are four such choices, each of which are equivalent because the cosine is an even function. I'll just multiply by the factor of 4, rename the external indices γ and γ', and rename the internal index α. This corresponds to the following Feynman diagram:



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Let's consider one such term.

$$\frac{1}{4}\sum_{\epsilon,\epsilon'=\pm}\left(\langle e^{i\epsilon\varphi_{\gamma,\alpha}(r_1)}e^{i\epsilon\varphi_{\alpha,\gamma'}(r_2)}\rangle - \langle e^{i\epsilon\varphi_{\gamma,\alpha}(r_1)}\rangle\langle e^{i\epsilon\varphi_{\alpha,\gamma'}(r_2)}\rangle\right) =$$
$$\frac{1}{2}\cos\left(\varphi^<_{\gamma,\alpha}(r_1)+\varphi^<_{\alpha,\gamma'}(r_2)\right)\left(\langle e^{i\varphi^>_{\gamma,\alpha}(r_1)}e^{i\varphi^>_{\alpha,\gamma'}(r_2)}\rangle - \langle e^{i\varphi^>_{\gamma,\alpha}(r_1)}\rangle\langle e^{i\varphi^>_{\alpha,\gamma'}(r_2)}\rangle\right)$$
$$+\frac{1}{2}\cos\left(\varphi^<_{\gamma,\alpha}(r_1)-\varphi^<_{\alpha,\gamma'}(r_2)\right)\left(\langle e^{i\varphi^>_{\gamma,\alpha}(r_1)}e^{-i\varphi^>_{\alpha,\gamma'}(r_2)}\rangle - \langle e^{i\varphi^>_{\gamma,\alpha}(r_1)}\rangle\langle e^{-i\varphi^>_{\alpha,\gamma'}(r_2)}\rangle\right) \quad \text{(B5)}$$

Here I just show that we expand the ϕ fields into fast and slow components and then take the expectation value with respect to the fast variables. The expectation values are pure real so we can then recombine the cosine of the slow variables as shown.

At this point I will take the liberty to say that we only want to consider the first term on the right hand side for this particular choice of α. The reason is that the expectation values connecting fields at r_1 and r_2 will constrain the difference between these positions to be $\mathcal{O}(1/\Lambda)$, which we'll see explicitly. It therefore makes sense to expand in $r = r_1 - r_2$, a relative coordinate. This will furthermore be sensible because the expectation values connecting r_1 and r_2 only depend on their relative position, r. We can naively see that expanding

$$\varphi^<_{\gamma,\alpha}(r_1)+\varphi^<_{\alpha,\gamma'}(r_1+r) = \phi^<_\gamma(r_1)+\phi^<_{\gamma+\pi}(r_1)-\phi^<_\alpha(r_1)-\phi^<_{\alpha+\pi}(r_1)$$
$$-\phi^<_\alpha(r_1+r)-\phi^<_{\alpha+\pi}(r_1+r)-\phi^<_{\gamma'}(r_1+r)-\phi^<_{\gamma'+\pi}(r_1+r) \quad \text{(B6)}$$

to leading order in r gives

$$\varphi^<_{\gamma,\alpha}(r_1)+\varphi^<_{\alpha,\gamma'}(r_1+r) \approx \varphi^<_{\gamma,\gamma'}(r_1)+\mathcal{O}(r). \quad \text{(B7)}$$

This is the general strategy. If we had considered one of the complex exponentials where $\epsilon \neq \epsilon'$ then we would have taken the cosine with the minus sign instead in order to perform the same expansion. We will not consider the behavior of the other term as it clearly generates a term that sums eight ϕ fields and therefore is generically less relevant than the original BCS term at tree level.

In order to perform this expansion, we have to normal order the cosine. This is done [9] by taking $\cos(\phi) =: \cos(\phi) : e^{-\frac{1}{2}\langle\phi^2\rangle}$ where the colons denote normal ordering. This will be an expectation value taken with respect to the *slow* fields inside the cosine. Note as well from the main text that

$$\langle e^{i\varphi^{>}_{\gamma,\alpha}(r_1)} e^{-i\varphi^{>}_{\alpha,\gamma'}(r_2)}\rangle = \exp\bigg(-\frac{1}{2}\langle(\varphi^{>}_{\gamma,\alpha}(r_1)+\varphi^{>}_{\alpha,\gamma'}(r_2))^2\rangle\bigg). \tag{B8}$$

If we combine the exponentiated expectation value from normal ordering with this factor, one finds that

$$\begin{aligned}\exp\bigg(-\frac{1}{2}\langle(\varphi^{<}_{\gamma,\alpha}(r_1)+\varphi^{<}_{\alpha,\gamma'}(r_2))^2\rangle\bigg)\exp\bigg(-\frac{1}{2}\langle(\varphi^{>}_{\gamma,\alpha}(r_1)+\varphi^{>}_{\alpha,\gamma'}(r_2))^2\rangle\bigg)& \\ = \exp\bigg(-\frac{1}{2}\langle(\varphi_{\gamma,\alpha}(r_1)+\varphi_{\alpha,\gamma'}(r_2))^2\rangle\bigg),&\end{aligned} \tag{B9}$$

This follows from the fact that $\langle\phi_\gamma\phi_{\gamma'}\rangle \propto \delta_{\gamma,\gamma'}$. We can now write out the first term of Eq. (B5) as

$$\frac{1}{2} : \cos\left(\varphi^{<}_{\gamma,\alpha}(r_1)+\varphi^{<}_{\alpha,\gamma'}(r_2)\right) : \; e^{-\frac{1}{2}\langle(\varphi_{\gamma,\alpha}(r_1)+\varphi_{\alpha,\gamma'}(r_2))^2\rangle}\left(1-e^{\langle\varphi^{>}_{\gamma,\alpha}(r_1)\varphi^{>}_{\alpha,\gamma'}(r_2)\rangle}\right). \tag{B10}$$

The problem now amounts to solving for these two expectation values. Let's consider the expectation value of the fast fields. First we use the fact that $\mathcal{L}_0$ is diagonal in γ to simplify:

$$\langle\varphi^{>}_{\gamma,\alpha}(r_1)\varphi^{>}_{\alpha,\gamma'}(r_2)\rangle = -\langle\phi^{>}_{\alpha}(r_1)\phi^{>}_{\alpha}(r_2)\rangle - \langle\phi^{>}_{\alpha+\pi}(r_1)\phi^{>}_{\alpha+\pi}(r_2)\rangle. \tag{B11}$$

We then use the same manipulations from Sec. III to find

$$\begin{aligned}\langle\phi^{>}_{\alpha}(r_1)\phi^{>}_{\alpha}(r_2)\rangle &= 2\pi\eta v l_\Lambda \int_{s\Lambda}^{\Lambda}\frac{d\omega\; dk_\parallel}{(2\pi)^2}\int_{-\Lambda}^{\Lambda}\frac{dk_\perp}{2\pi}\frac{e^{i(k\cdot x+\omega\tau)}}{\omega^2+v^2k_\parallel^2} \\ &\approx 2\pi\eta v\;\delta_\Lambda(x^\alpha_\perp)\int_{s\Lambda}^{\Lambda}\frac{d\omega\; dk_\parallel}{(2\pi)^2}\;\frac{\cos(k_\parallel x_\parallel+\omega\tau)}{\omega^2+v^2k_\parallel^2} \\ &= 2\pi\eta\;\delta_\Lambda(x^\alpha_\perp)\int_{s\Lambda}^{\Lambda}\frac{dq}{(2\pi)^2|q|}\int_{-\pi}^{\pi}d\theta\cos(qr_\alpha\cos(\theta)) \\ &= \eta\;\delta_\Lambda(x^\alpha_\perp)\int_{s\Lambda}^{\Lambda}\frac{dq}{|q|}J_0(qr_\alpha)\end{aligned} \tag{B12}$$

I did a few things there. In the first line we took

$$l_\Lambda \int_{-\Lambda}^{\Lambda} \frac{dk_\perp}{2\pi}\ e^{i(k\cdot x+\omega\tau)} = e^{i(k_\parallel x_\parallel+\omega\tau)}\left(\frac{l_\Lambda \sin(\Lambda x_\perp)}{x_\perp}\right) \approx e^{i(k_\parallel x_\parallel+\omega\tau)}\delta_\Lambda(x_\perp) \tag{B13}$$

where δ_Λ is defined in Eq. (19). This essentially fixes $x_\perp \sim 1/\Lambda$. I then changed variables to $q = \sqrt{k_\parallel^2 + \omega^2/v^2}$ and defined the cosine in terms of $r_\alpha = \sqrt{x_\parallel^2 + v^2\tau^2}$. Note that this variable r_α is the magnitude of the *relative* coordinate, $r_1 - r_2$. For s very close to 1 we can approximate the integral and obtain

$$\langle \phi^>_\alpha(r_1)\phi^>_\alpha(r_2)\rangle \approx -\eta \ln(s) J_0(\Lambda r_\alpha)\delta_\Lambda(x^\alpha_\perp). \tag{B14}$$

If we define $s = 1-\nu$ where $0 < \nu \ll 1$, this gives us

$$\left(1 - e^{\langle \varphi^>_{\gamma,\alpha}(r_1)\varphi^>_{\alpha,\gamma'}(r_2)\rangle}\right) \approx -2\eta\nu J_0(\Lambda r_\alpha)\delta_\Lambda(x^\alpha_\perp). \tag{B15}$$

The Bessel function is peaked near $r_\alpha = 0$, motivating us to expand in small r_α (which should be $\sim 1/\Lambda$).

Now we come to the other expectation value, which ostensibly looks like a problem. Expanding, we get

$$\begin{aligned}\langle(\varphi_{\gamma,\alpha}(r_1) + \varphi_{\alpha,\gamma'}(r_2))^2\rangle = \langle\phi^2_\gamma(r_1)\rangle + \langle\phi^2_{\gamma+\pi}(r_1)\rangle + \langle\phi^2_{\gamma'}(r_2)\rangle + \langle\phi^2_{\gamma'+\pi}(r_2)\rangle \\ + \langle(\phi_\alpha(r_1) - \phi_\alpha(r_2))^2\rangle + \langle(\phi_{\alpha+\pi}(r_1) - \phi_{\alpha+\pi}(r_2))^2\rangle.\end{aligned} \tag{B16}$$

We already computed the α-dependent terms, approximating the exponentiated result in Eq. (18). The problem is that the expectation value of ϕ^2 has a non-integrable divergence for small momenta. Inserting a finite system size,

$$\begin{aligned}-\frac{1}{2}\langle\phi_\gamma(r_1)^2\rangle &= -\pi\eta v l_\Lambda \int \frac{d^2k d\omega}{(2\pi)^3}\frac{1}{\omega^2 + v^2k_\gamma^2} \\ &= -\frac{1}{2}\eta\int_{1/L}^{\Lambda}\frac{dk_\gamma}{|k_\gamma|} \\ &= -\frac{1}{2}\eta\ln(\Lambda L).\end{aligned} \tag{B17}$$

This would imply that the relevant term from Eq. (B5) takes the form

$$-\eta\nu\ (\Lambda L)^{-2\eta}\ \delta_\Lambda(x^\alpha_\perp)\ J_0(\Lambda r_\alpha)\ G^0_\alpha(r)\ G^0_{\alpha+\pi}(r)\ :\cos\left(\varphi^<_{\gamma,\alpha}(r_1) + \varphi^<_{\alpha,\gamma'}(r_2)\right): \tag{B18}$$

where I refer to the result of Eq. (18) as $G^0_\alpha(r)$ and $r = r_1 - r_2$ is the relative coordinate. It's actually the case that $G^0_\alpha(r) = G^0_{\alpha+\pi}(r)$, so this could be simplified further. This term

looks like it vanishes upon taking $L \to \infty$. Proceeding as if everything's fine, however, we can now expand the cosine as prefaced earlier. The leading-order contribution gives us back the $\cos(\varphi^{<}_{\gamma,\gamma'})$ of the original interaction. Only that's not *exactly* what it gives back – what we really have is $:\cos(\varphi^{<}_{\gamma,\gamma'}):$, which we can't compare directly to the original term. We therefore have to invert the relation we used to normal-order the cosine. Using the fact that $\mathcal{L}_0$ is diagonal in the patch fields, we have that

$$e^{-\frac{1}{2}\langle(\varphi^{<}_{\gamma,\gamma'}(r_1))^2\rangle} = e^{-2\langle(\phi^{<})^2\rangle} = (\Lambda' L)^{-2\eta}. \tag{B19}$$

This means that the factor we are absorbing is precisely the factor that would have caused the entire result to vanish! Combining, we obtain the familiar factor $(\Lambda L)^{-2\eta}(\Lambda' L)^{2\eta} = s^{2\eta}$.

Changing variables of integration $\int d^3r_1 \int d^3r_2 \to \int d^3r_1 \int d^3r$ so that we integrate over the relative coordinate, we find that the net contribution to $I^{(2)}$, which I'll call $I^{(2a)}$ because we threw some terms away, is

$$I^{(2a)} \sim \frac{1}{N^2}\sum_{\gamma,\gamma'}\frac{1}{N^2}\sum_{\alpha} g_{\text{BCS}}(\gamma-\alpha)g_{\text{BCS}}(\alpha-\gamma')\int d^3r_1 \int d^3r \times \\ \left(-\eta\nu\ s^{2\eta}\ \delta_\Lambda(x^\alpha_\perp)\ J_0(\Lambda r_\alpha)\ \big(G^0_\alpha(r)\big)^2\ \cos\big(\varphi^{<}_{\gamma,\gamma'}(r_1)\big)\right). \tag{B20}$$

Rearranging,

$$I^{(2a)} \sim -\eta\nu\ s^{2\eta}\ \frac{1}{N^2}\sum_{\gamma,\gamma'}\left(\int d^3r_1\ \cos\big(\varphi^{<}_{\gamma,\gamma'}(r_1)\big)\right)\times \\ \left(\frac{1}{N^2}\sum_{\alpha} g_{\text{BCS}}(\gamma-\alpha)g_{\text{BCS}}(\alpha-\gamma')\left(\int d^3r\ \delta_\Lambda(x^\alpha_\perp)\ J_0(\Lambda r_\alpha)\ \big(G^0_\alpha(r)\big)^2\right)\right). \tag{B21}$$

The first thing to note is that there's an extra factor of $1/N$ because we got rid of the second sum with a delta function. This won't be a problem, however, because the integral over $x^\alpha_\perp$ gives a factor of $2/\Lambda$ due to the smeared out delta function. Recall from the definition of N that $1/N\Lambda = 1/\pi k_B$. The remaining terms in the integral are functions of $\Lambda x_\parallel$ and $\Lambda v\tau$, so we can rescale the integral and obtain a constant times $1/v\Lambda^2$. Interestingly it doesn't seem like I'll be able to get rid of this factor... I'm not sure if this is correct. We can now define a positive constant C where

$$C = \Lambda^2 v \int d^2r\ J_0(\Lambda r_\gamma)\big(G^0_\gamma(r)\big)^2 \tag{B22}$$

is independent of Λ and v as discussed (and is therefore invariant under rescaling). Rescaling the r_1 spacetime dimensions gives a factor of s^{-2}. We collect terms to find

$$I^{(2a)} \sim -\frac{\eta\nu C}{\Lambda^2 v}\ s^{-2+2\eta}\ \frac{1}{N^2}\sum_{\gamma,\gamma'}\left(\frac{1}{N}\sum_{\alpha} g_{\mathrm{BCS}}(\gamma-\alpha)g_{\mathrm{BCS}}(\alpha-\gamma')\right)\left(\int d^3r_1\ \cos\left(\varphi^{<}_{\gamma,\gamma'}(r_1)\right)\right). \tag{B23}$$

We have essentially recovered the desired form. Define $\tilde{C} = A\frac{\eta C}{\Lambda^2 v} > 0$ for convenience where A denotes any numerical factors I've ignored up until now. Note in our definition of an RG time, $s = e^{-t}$ where $t \ll 1$, we would identify that $1 - s = \nu = t$. The general equation for the coupling constant flow under RG is then

$$\begin{aligned} g_{\mathrm{BCS}}(t,\gamma-\gamma') - g_{\mathrm{BCS}}(0,\gamma-\gamma') &= e^{(2-2\eta)t}\ g_{\mathrm{BCS}}(0,\gamma-\gamma') \\ &\quad - e^{(2-2\eta)t}\tilde{C}t\int\frac{d\alpha}{2\pi}\ g_{\mathrm{BCS}}(0,\gamma-\alpha)g_{\mathrm{BCS}}(0,\alpha-\gamma') \end{aligned} \tag{B24}$$

Note that I've taken $N \to \infty$ and turned the sum over α into a product. What is typically done now is to rewrite the functions $g_{\mathrm{BCS}}(t,\gamma)$ as an infinite sum of harmonics so that we can turn the convolution into a product. The harmonics are defined as

$$g^l_{\mathrm{BCS}} = \int\frac{d\gamma}{2\pi}\ \cos(l\gamma)\ g_{\mathrm{BCS}}(\gamma) \tag{B25}$$

where $l \in 2\mathbb{Z}$ due to Bose symmetry. The above equation now takes the form

$$g^l_{\mathrm{BCS}}(t) = e^{(2-2\eta)t}\ \left(g^l_{\mathrm{BCS}}(0) - \tilde{C}t\big(g^l_{\mathrm{BCS}}(0)\big)^2\right). \tag{B26}$$

Taking the derivative of both sides with respect to t yields

$$\frac{g^l_{\mathrm{BCS}}(t)}{dt} = (2-2\eta)e^{(2-2\eta)t}\ \left(g^l_{\mathrm{BCS}}(0) - \tilde{C}t\big(g^l_{\mathrm{BCS}}(0)\big)^2\right) - \tilde{C}e^{(2-2\eta)t}\big(g^l_{\mathrm{BCS}}(0)\big)^2. \tag{B27}$$

Now using the definition of $g^l_{\mathrm{BCS}}(t)$, we plug that into the above equation to remove all dependence of $g^l_{\mathrm{BCS}}(0)$. The resulting equation is then expanded to linear order in t, which is assumed to be infinitesimal. Discarding terms $\mathcal{O}(t)$ yields the beta function for the harmonics,

$$\frac{dg^l_{\mathrm{BCS}}}{dt} = (2-2\eta)g^l_{\mathrm{BCS}} - \tilde{C}(g^l_{\mathrm{BCS}})^2. \tag{B28}$$

Appendix C: Obtaining the Dual Lagrangian

I found it easiest to begin thinking about this by considering a multi-component Luttinger liquid Hamiltonian,

$$\mathcal{H} = \frac{u}{2\pi}\sum_{\gamma,\gamma'}\left(\frac{1}{K}(\nabla_\gamma\phi_\gamma)\big(\delta_{\gamma,\gamma'} + f^{\phi}_{\gamma,\gamma'}\big)(\nabla_{\gamma'}\phi_{\gamma'}) + K(\nabla_\gamma\theta_\gamma)\big(\delta_{\gamma,\gamma'} + f^{\theta}_{\gamma,\gamma'}\big)(\nabla_{\gamma'}\theta_{\gamma'})\right). \quad \text{(C1)}$$

Note that this isn't intended to be equivalent to $\mathcal{L}_0$ – the argument should be general. This Hamiltonian is of the form

$$\mathcal{H} = \sum_{\gamma,\gamma'}\big((\nabla_\gamma\phi_\gamma)M^{\phi}_{\gamma,\gamma'}(\nabla_{\gamma'}\phi_{\gamma'}) + (\nabla_\gamma\theta_\gamma)M^{\theta}_{\gamma,\gamma'}(\nabla_{\gamma'}\theta_{\gamma'})\big). \quad \text{(C2)}$$

There are two ways to write out the action [9] depending on which fields we'll want to integrate out:

$$S_\phi = \int_0^\beta d\tau\int d^2x\sum_{\gamma,\gamma'}\left(-i\frac{1}{\pi}(\nabla_\gamma\theta_\gamma)(\partial_\tau\phi_\gamma)\delta_{\gamma,\gamma'} + \mathcal{H}\right) \quad \text{(C3)}$$

$$S_\theta = \int_0^\beta d\tau\int d^2x\sum_{\gamma,\gamma'}\left(i\frac{1}{\pi}(\nabla_\gamma\phi_\gamma)(\partial_\tau\theta_\gamma)\delta_{\gamma,\gamma'} + \mathcal{H}\right) \quad \text{(C4)}$$

When considering S_ϕ, as I've suggestively defined it, we would naturally want to integrate out the θ_γ fields; vice versa for S_θ. In order to do this, we want to complete the square. Take S_ϕ as an example. Considering just the relevant terms, we have

$$\sum_{\gamma,\gamma'}(\nabla_\gamma\theta_\gamma)M^{\theta}_{\gamma,\gamma'}(\nabla_{\gamma'}\theta_{\gamma'}) - \frac{i}{\pi}(\nabla_\gamma\theta_\gamma)(\partial_\tau\phi_{\gamma'})\delta_{\gamma,\gamma'}. \quad \text{(C5)}$$

This is an equation of the form $x^TMx - 2b^Tx$ where $M = M^T$ is a symmetric matrix. This can be rewritten as

$$x^TMx - 2b^Tx = (x - M^{-1}b)^TM(x - M^{-1}b) - b^TM^{-1}b, \quad \text{(C6)}$$

where we have completed the square. The first term can now be integrated out because, in the exponentiated action e^{-S} that appears in the path integral, it is a simple Gaussian integral. The effective action now contains a term coupling $\partial_\tau\phi_\gamma$ on different patches, and the matrix connecting these patches is given by $(M^\theta)^{-1}$ times some constants (which we absorbed into our definition of b). Specifically, we find that

$$S_\phi = \int_0^\beta d\tau\int d^2x\left(\frac{1}{4\pi^2}(\partial_\tau\phi_\gamma)[(M^\theta)^{-1}]_{\gamma,\gamma'}(\partial_\tau\phi_{\gamma'}) + (\nabla_\gamma\phi_\gamma)M^{\phi}_{\gamma,\gamma'}(\nabla_{\gamma'}\phi_{\gamma'})\right) \quad \text{(C7)}$$

$$S_\theta = \int_0^\beta d\tau \int d^2x \left(\frac{1}{4\pi^2} (\partial_\tau \theta_\gamma)[(M^\phi)^{-1}]_{\gamma,\gamma'} (\partial_\tau \theta_{\gamma'}) + (\nabla_\gamma \theta_\gamma) M^\theta_{\gamma,\gamma'} (\nabla_{\gamma'} \theta_{\gamma'}) \right) \tag{C8}$$

Given this relationship, it is straightforward to switch between the effective Lagrangian $\mathcal{L}_0$ and the dual Lagrangian by just inverting the matrix coupling each sector (charge or momentum). The last thing we need to know is a trick given in the paper:

$$\left(\delta_{\gamma,\gamma'} + \frac{1}{N} f_\rho \right)^{-1} = \delta_{\gamma,\gamma'} - \frac{1}{N} \frac{f_\rho}{1 + f_\rho}. \tag{C9}$$

Given this, the dual Lagrangians are

$$\mathcal{L}[\phi] = \frac{k_B}{4\pi N \eta} \sum_{\gamma\gamma'} \phi_\gamma \left(v^{-1} \omega^2 \big(\delta_{\gamma\gamma'} + \frac{1}{N} f_\rho \big) - v k_\gamma k_{\gamma'} \big(\delta_{\gamma\gamma'} + \frac{1}{N} f_j \big) \right) \phi_{\gamma'}, \tag{C10}$$

$$\mathcal{L}[\theta] = \frac{k_B \eta}{4\pi N} \sum_{\gamma,\gamma'} \theta_\gamma \left(v^{-1} \omega^2 \big(\delta_{\gamma\gamma'} + \frac{1}{N} \tilde{f}_\rho \big) - v k_\gamma k_{\gamma'} \big(\delta_{\gamma\gamma'} + \frac{1}{N} \tilde{f}_j \big) \right) \theta_{\gamma'}, \tag{C11}$$

where we define dual Landau parameters

$$\tilde{f}_\rho = -\frac{f_j}{1 + f_j} \qquad \tilde{f}_j = -\frac{f_\rho}{1 + f_\rho}. \tag{C12}$$

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[11] It is actually not quite accurate to say that $\mathcal{L}_0$ comes just from the free part of the Lagrangian. There are also intra-patch backscattering processes that are not kinematically suppressed by the curvature of the Bose surface. I assume it is because of complications like this that the authors do not relate the phenomenological parameters to the microscopic parameters..

[12] This is explained differently in the text. I found their explanation pretty opaque, but this is what made sense to me. It also reflects the fact that the same thing will be true in the $3+1$D BLL.

[13] The authors claim, more generally, that ϕ_γ^- cannot be gapped out by a pairing operator because it is not charged under the microscopic U(1). While I'm sure that this is true, I don't understand the logic.

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