A Exam Question 2: Ultrasound Attenuation in Sr_2RuO_4

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I. INTRODUCTION

Sound waves in solids are periodic and propagating deformations of the atomic lattice. The speed of sound is related to the potential energy pulling the atoms back to their equilibrium positions. As a sound wave propagates, its amplitude will decay as dissipative processes transfer energy into heat. We define the attenuation constant of a sound wave, α, as half the rate of its decay [1]. We can therefore write down a solution to the wave equation,

$$\vec{u} = \vec{u}_0 \exp\left(i\omega t - i\vec{q}\cdot\vec{r} - \frac{1}{2}\alpha\hat{q}\cdot\vec{r}\right) \tag{1}$$

where $\vec{u}$ is the velocity of the atoms at position $\vec{r}$ and time t. The sound velocity is given by the ratio $v_s = \omega/|\vec{q}|$ as sound waves are linearly-dispersing at low energies.

There are a variety of means by which sound can dissipate in a solid, but many of these have trivial temperature dependences and therefore will not concern us here [1]. The primary dissipation channels we will consider are electron-phonon scattering and electron-impurity scattering. In general one can think of these working in tandem to transfer heat from the wave to an external reservoir, and indeed this is often the way expressions for the sound attenuation constant are derived [2]. We will consider the problem in a more abstract form, but it is useful to have this method in mind.

The question we have in mind is how does the ultrasound attenuation constant vary with temperature across the superconducting transition. The canonical prediction put forth by BCS in Ref. [3] was one of the first indications of their success in describing superconductivity. They found that the ultrasound attenuation constant should drop precipitously as one enters the superconducting state, and this is generally what one finds in s-wave "conventional" superconductors. We will derive this result in Sec. II. This behavior cannot necessarily be imported into the context of non-s-wave pairing, however. Measurements some heavy-fermion materials, namely UBe_{13} and UPt_3, have shown a peak in the longitudinal attenuation factor upon crossing T_c [4, 5]. More recently, Brad's group measured a similar peak below T_c in Sr_2RuO_4 [6]. What we will investigate is whether the peak in the ultrasound attenuation is a signature of a sign-changing (triplet) gap. We will show that the peak is not a *signature* of the gap, but it is *possible* that a triplet gap could lead to such a peak. We will conclude with an evaluation of an alternative explanation provided by Coffey in Ref. [7, 8].

II. S-WAVE ATTENUATION

The first question to ask is what is different about life below T_c. At the mean-field level, which will be the basis of this discussion, a superconducting gap opens up and the Hamiltonian is diagonalized by coherent superpositions of single-particle states. The gap inhibits scattering processes such that, at adequately low temperatures, we would expect the sound attenuation to decrease exponentially. What happens precisely at the transition, however, is not yet clear.

Following along with Refs. [3, 9], we will treat the problem of ultrasound attenuation at the level of Fermi's golden rule. Thus we will assume that the sound wave can be thought of as a number-conserving perturbation to the Hamiltonian of the form

$$\mathcal{H}_{\text{int}} = \sum_{k_1,k_2} \sum_{\sigma} M_{k_1,k_2} c^{\dagger}_{k_1,\sigma} c_{k_2,\sigma}, \tag{2}$$

and we will simply ask what the total scattering rate is in the system. Note that we will only treat interactions that do not flip the spin, as that will be adequate for the purpose of ultrasound attenuation; the more general spin-dependent problem is discussed in Appendix A. It will be assumed that this scattering rate is directly proportional to the sound attenuation constant, $\alpha(T)$. The statement of Fermi's golden rule is

$$\Gamma(\omega, T) = \frac{2\pi}{\hbar} \int d^3k_1 \int d^3k_2 \; |\langle \mathcal{H}_{\text{int}} \rangle|^2 \big(f_{k_1}(1 - f_{k_2}) - f_{k_2}(1 - f_{k_1}) \big) \delta(E_1 - E_2 - \hbar\omega) \tag{3}$$

where f_k is the Fermi-Dirac distribution function. This is intentionally schematic, as the true nature of this argument is about phase space and interference between quasiparticles. We will end up dividing by Γ_N, the scattering rate in the normal state, to remove arbitrary constants.

If we had access to the matrix elements M_{k_1,k_2}, we would have enough information to do this simplified calculation for a normal metal (by "normal metal" I mean that the unperturbed Hamiltonian $\mathcal{H}_0$ is diagonal in the single-particle basis). The reason is that each transition in Eq. (2) can be thought of as independent of every other transition. Thus we just square the matrix element, insert the dispersion for $E_{1/2}$, and plug and chug. In a superconductor, however, this is no longer valid. Specifically the unperturbed Hamiltonian $\mathcal{H}_0$ is *not* diagonal in the basis of $c_{k,\sigma}$ due to the BCS pairing operator. The BCS mean field Hamiltonian is diagonal in the basis of Bogoliubov quasiparticles, $\gamma_{k,\tau}$, which are composed of coherent superpositions of $c_{k,\sigma}$ states. We must therefore rewrite the interaction

Hamiltonian in the form

$$\mathcal{H}_{\text{int}} = \sum_{k_1,k_2}\sum_{\tau_1,\tau_2}\left(M^{\gamma,1}_{k_1,k_2}\gamma^\dagger_{k_1,\tau_1}\gamma_{k_2,\tau_2} + M^{\gamma,2}_{k_1,k_2}\gamma^\dagger_{k_1,\tau_1}\gamma^\dagger_{k_2,\tau_2} + h.c.\right), \tag{4}$$

in which case we can treat the matrix elements independently as we would for the normal metal.

Going from the matrix elements M to M^γ is straightforward because the $c_{k,\sigma}$ operators can be written as a superposition of $\gamma_{k,\tau}$ operators. Let's first set up the formalism to do that. The mean field Hamiltonian reads

$$\mathcal{H}_{MF} = \sum_{k,\sigma}\xi_k c^\dagger_{k,\sigma}c_{k,\sigma} - \Delta\sum_k\left(c_{k\uparrow}c_{-k,\downarrow} + c^\dagger_{-k,\downarrow}c_{k,\uparrow} + \dots\right) \tag{5}$$

where we use the standard notation that $\xi_k = \epsilon_k - \mu$ and the gap Δ is proportional to the pairing operator, $\langle c_{k,\sigma}c_{-k,-\sigma}\rangle$ (I've ignored constant terms). Naturally the gap must satisfy a self-consistency condition. We diagonalize $\mathcal{H}_{MF}$ with a Bogoliubov transformation

$$\begin{aligned} c_{k,\uparrow} &= u^*_k\gamma_{k,0} + v_k\gamma^\dagger_{-k,1} \\ c^\dagger_{-k,\downarrow} &= -v^*_k\gamma_{k,0} + u_k\gamma^\dagger_{-k,1}. \end{aligned} \tag{6}$$

where the coefficients satisfy $|u_k|^2 + |v_k|^2 = 1$. They coefficients are found to be

$$u_k = \sqrt{\frac{1}{2}\left(1 + \frac{\xi^2}{E_k^2}\right)} \qquad v_k = \sqrt{\frac{1}{2}\left(1 - \frac{\xi^2}{E_k^2}\right)} \tag{7}$$

and the energy eigenvalue is $E_k = \sqrt{\xi_k^2 + \Delta^2}$.

As has been made explicit in Eq. (6), we see that a Bogoliubov quasiparticle is a superposition of a state at (k,σ) and a state at $(-k,-\sigma)$. Thus the scattering operator $c^\dagger_{k,\sigma}c_{k',\sigma}$ connects the same Bogoliubov quasiparticle states as the operator $c^\dagger_{-k',-\sigma}c_{-k,-\sigma}$. Furthermore, the quasiparticle scattering terms obtained by expanding these terms will not be found in any other expansion of single-particle scattering operators. Thus, considering pairs of scattering operators such as these will allows us to convert from the M to the M^γ matrix elements.

At this stage, it should be clear that we have to make an assumption about the matrix elements. Specifically we will assume that $M_{k,k'} = \pm M_{-k',-k}$. This is a reasonably broad assumption: we are simply assuming that the interaction term is even or odd under time reversal (technically this would also involve spin – see Appendix A). This creates two possible cases we will want to consider. Factoring out the magnitude of the matrix element, we'll want to find the sum $c^\dagger_{k,\sigma} c_{k',\sigma} + \alpha\, c^\dagger_{-k',-\sigma} c_{-k,-\sigma}$ where $\alpha = 1$ denotes Case I interactions and $\alpha = -1$ denotes Case II interactions (to use the terminology of Ref. [3]). Expanding the sum, we find

$$c^\dagger_{k,\sigma} c_{k',\sigma} + \alpha\, c^\dagger_{-k',-\sigma} c_{-k,-\sigma} = (u_{k_1} u_{k_2} - \alpha v_{k_1} v_{k_2})\big(\gamma^\dagger_{k',\sigma}\gamma_{k,\sigma} + \alpha\, \gamma^\dagger_{-k,-\sigma}\gamma_{-k',-\sigma}\big) + (u_{k_1} v_{k_2} + \alpha u_{k_2} v_{k_1})\big(\gamma^\dagger_{k',\sigma}\gamma^\dagger_{-k,-\sigma} + \alpha\, \gamma_{-k',-\sigma}\gamma_{k,\sigma}\big) \quad (8)$$

where I've adopted the convention used in Ref. [9] for the spin indices on the γ operators. Note that there is one prefactor for the quasiparticle scattering terms ($\gamma^\dagger\gamma$) and another for the creation/annihilation terms. These prefactors are what we'll refer to as the coherence factors. Multiplying the matrix elements M by the coherence factors takes us to the matrix elements M^γ. Physically this corresponds to the fact that an operator which scatters single electrons will end up scattering multiple Bogoliubov quasiparticles. The quasiparticle scattering processes can add constructively or destructively, resulting in the two different coherence factors. We can simplify the coherence factors using the definitions of u_k and v_k:

$$(u_{k_1} u_{k_2} - \alpha v_{k_1} v_{k_2})^2 \to \frac{1}{2}\left(1 - \alpha\frac{\Delta^2}{E_1 E_2}\right) \qquad (u_{k_1} v_{k_2} + \alpha u_{k_2} v_{k_1})^2 \to \frac{1}{2}\left(1 + \alpha\frac{\Delta^2}{E_1 E_2}\right). \quad (9)$$

Note in the above that we ignore a term proportional $\xi_1\xi_2/E_1E_2$ in both coherence factors because it will integrate to zero ($\xi(k)$ has been linearized about the Fermi surface, so it is an odd function over our range of integration).

In principle the decay rate Γ, defined in Eq. (3), is a function of both frequency and temperature. The sound attenuation coefficient is related to the $\omega \to 0$ limit of $\Gamma(\omega, T)$, so that is the limit we will consider. In order to get a new Fermi's golden rule expression for Γ, we are now taking the expectation value of $\mathcal{H}_{\text{int}}$ with respect to the mean-field ground state. This means that we will have both scattering terms and creation/annihilation terms to consider. These contributions differ in their coherence factors and in the product of Fermi factors that accompany them:

$$\Gamma(\omega, T) = \int_{-\infty}^{\infty} dE_1\, dE_2 \left(|M|^2 N_s(E_1) N_s(E_2)\left(1 - \alpha\frac{\Delta^2}{E_1E_2}\right)\big(f(E_1) - f(E_2)\big)\delta(E_1 - E_2 - \hbar\omega) + |M|^2 N_s(E_1) N_s(E_2)\left(1 + \alpha\frac{\Delta^2}{E_1E_2}\right)\big(1 - f(E_1) - f(E_2)\big)\delta(E_1 + E_2 + \hbar\omega)\right) \quad (10)$$

There is an additional subtlety here. As we are considering scattering processes in the limit $\omega \to 0$, certain scattering processes will not contribute. These can be thought of as processes that connect quasiparticle states above and below the superconducting gap, as they would require $\hbar\omega > 2\Delta$. We therefore only consider scattering processes where $\text{sign}(E_1) = \text{sign}(E_2)$ and creation/annihilation processes where $\text{sign}(E_1) = -\text{sign}(E_2)$. These correspond to the first and second terms in Eq. (10), respectively. The density of states is $N_s(E) = N(0)|E|/\sqrt{E^2 - \Delta^2}$ is zero for $|E| < \Delta$, and $N(0)$ is the electron density of states at the Fermi level. We can actually combine the two terms in Eq. (10) by taking $E_1 > 0$ and making the sign of E_2 explicit. This simplification was pointed out by BCS in a footnote in their original paper [3]. Evaluating the delta functions, and noting $1 - f(-E) = f(E)$, then yields

$$\Gamma(\omega, T) = \int_\Delta^\infty dE\ N_s(E) N_s(E + \hbar\omega)|M|^2 \left(1 - \alpha \frac{\Delta^2}{E(E + \hbar\omega)}\right)\big(f(E) - f(E + \hbar\omega)\big) \quad (11)$$

where we ignore constant prefactors. The prefactors will be irrelevant because we will take $|M|^2$ to be a constant and divide the result of Eq. (11) by the normal state value of the scattering rate,

$$\Gamma_N(\omega, T) = N(0)^2|M|^2 \int_0^\infty dE\ \big(f(E) - f(E + \hbar\omega)\big) \to \frac{\hbar\omega}{2} N(0)^2|M|^2, \quad (12)$$

where in the last line we took the small-ω limit.

We're now in a position to assess the temperature dependence of the two cases. The temperature dependence of the gap is given by $\Delta(T) = \Delta_0\sqrt{1 - T/T_c}$ [9], but the specific temperature dependence will not concern us here.

1. For a Case I process we take $\alpha = 1$, which is to say that the interaction is time-reversal invariant. As $\omega \to 0$ we can see that the coherence factor vanishes as $E \to \Delta$, which is exactly where the density of states diverges. The product of these terms is no longer peaked at $E = \Delta$. Thus, destructive interference between quasiparticles negates the diverging density of states near the superconducting gap. As we cross the transition temperature the lower bound of the integral increases from zero, resulting in a sharp decrease in the scattering rate. This is shown in Fig. 1.

2. For a Case II process, $\alpha = -1$ and the interaction breaks time-reversal symmetry. Now we have constructive interference where we previously had destructive interference. When the gap opens up, the density of states near Δ diverges and the scattering rate *increases*. Γ reaches a peak at some finite temperature and then decrease exponentially

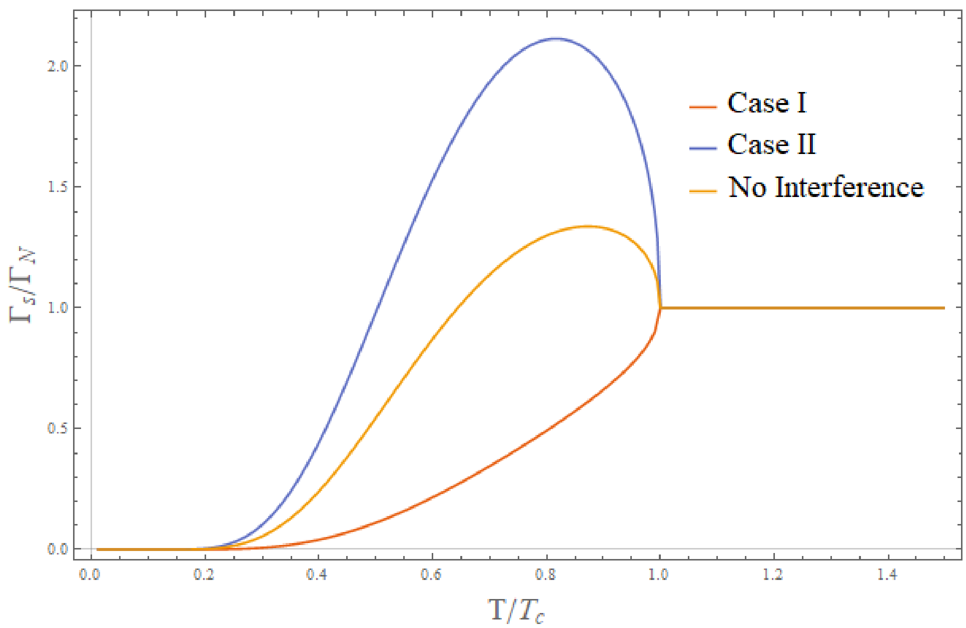

FIG. 1. Plot of the ratio of the superconducting scattering rate to the normal state scattering rate, Γ_s/Γ_N, as a function of temperature across the superconducting transition. Both cases are shown, as well as the result if interference terms were neglected. A cutoff above Δ was used to obtain finite results for the latter two curves, so the height and width of the peaks is somewhat arbitrary.

as $T \to 0$ and scattering processes are gapped out. It's worth noting that the Case II integral is actually formally infinite due to the divergence as $E \to \Delta$, so one must either use a phenomenological cutoff or assume that the density of states is disorder-broadened to obtain finite results [9].

In the context of these results, longitudinal sound attenuation is the archetypal Case I process. The electron-phonon and (non-magnetic) electron-impurity scattering processes responsible for sound attenuation are spin-independent and time-reversal symmetric [2]. Transverse sound attenuation can involve more complicated processes that we do not consider here. The success of this simple calculation in describing the sharp decrease in ultrasonic attenuation was one of the early successes of BCS theory [3, 10]. The peak seen in Case II processes has also been verified experimentally, and in the context of nuclear spin relaxation it is referred to as the Hebel-Slichter peak [11].

III. P-WAVE ATTENUATION

S-wave superconductivity involved pairing states at (k,σ) and $(-k,-\sigma)$ in a spin singlet. This is to say, if one were to decompose the expectation value of the pairing operator into a spin and momentum component, the spin component was antisymmetric:

$$b^k_{\sigma,\sigma'} = \langle c_{k,\sigma} c_{-k,\sigma'} \rangle = \phi(k)\chi_{\sigma,\sigma'} \tag{13}$$

with $\chi_{\sigma,\sigma'} = -\chi_{\sigma',\sigma}$ and $\phi(k)$ a constant. This need not be the case, however. The relation $b^k_{\sigma,\sigma'} = -b^{-k}_{\sigma',\sigma}$ must hold due to anticommutation relations, but that can also be satisfied with a symmetric function $\chi_{\sigma,\sigma'}$ and an orbital function that changes sign, $\phi(k) = -\phi(-k)$ (as well as other spin singlet arrangements with higher angular momentum). We'll now generalize the above formalism to include triplet pairing states. The gap function is defined as

$$\Delta^k_{\sigma,\sigma'} = -\sum_{k'} \sum_{\tau,\tau'} V^{k,k'}_{\sigma,\sigma',\tau,\tau'} b^{k'}_{\tau,\tau'} \tag{14}$$

where V is the interaction in the BCS pairing channel. We will not concern ourselves with the details of the interaction, presuming instead that the momentum and spin dependence of the gap is known. The gap has two spin-1/2 indices and can therefore be expressed as a 2×2 matrix in spin space:

$$\hat{\Delta}_k = \begin{pmatrix} \Delta_k^{\uparrow\uparrow} & \Delta_k^{\uparrow\downarrow} \\ \Delta_k^{\downarrow\uparrow} & \Delta_k^{\downarrow\downarrow} \end{pmatrix}. \tag{15}$$

Given this form, we'll express the gap matrix as a product of Pauli matrices. The gap is generally characterized by the vector $\vec{d}(k)$, in terms of which $\hat{\Delta}_k = i(\vec{d}(k)\cdot\hat{\vec{\sigma}})\hat{\sigma}_y$. The mean-field Hamiltonian can be expressed in terms of the gap, as in Eq. (5), and diagonalized by a Bogoliubov transformation. Again we define Bogoliubov quasiparticles that are related to the electon and hole states by

$$\begin{pmatrix} c_{k,\uparrow} \\ c_{k,\downarrow} \\ c^\dagger_{-k,\uparrow} \\ c^\dagger_{-k,-\downarrow} \end{pmatrix} = \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}^*_{-k} & \hat{u}^*_{-k} \end{pmatrix} \begin{pmatrix} \gamma_{k,\uparrow} \\ \gamma_{k,\downarrow} \\ \gamma^\dagger_{-k,\uparrow} \\ \gamma^\dagger_{-k,-\downarrow} \end{pmatrix}. \tag{16}$$

The parameters in the unitary transformation are themselves matrices in spin space,

$$\hat{u}_k = \frac{E_k + \xi_k}{\sqrt{2E_k(E_k+\xi_k)}}\hat{\sigma}_0 \qquad \hat{v}_k = \frac{-\hat{\Delta}_k}{\sqrt{2E_k(E_k+\xi_k)}}, \tag{17}$$

where we define $E_k = \sqrt{\xi_k + |\Delta_k|^2}$ and $|\Delta_k|^2 = \frac{1}{2}\mathrm{Tr}\hat{\Delta}_k\hat{\Delta}_k^\dagger$. Note that we have assumed a unitary gap in the above, which means $\hat{\Delta}_k\hat{\Delta}_k^\dagger \propto \hat{\sigma}_0$.

Given this formalism, it is straightforward to pick different triplet pairing states and extract the coherence factors. The most general triplet pairing state will have $\vec{d}(k) = d_x(k)\hat{x} + d_y(k)\hat{y} + d_z(k)\hat{z}$ where each component of $\vec{d}$ can be complex (of course, not all of these choices make for unitary gaps). The direction $\hat{d}$ corresponds to different pairing functions $\chi_{\sigma,\sigma'}$ and the magnitude of $|\vec{d}(\vec{k})|$ is the magnitude of the gap at momentum $\vec{k}$ (which is presumed to lie on the Fermi surface). We can then solve the problem generally via the Bogoliubov transformation

$$\begin{pmatrix} c_{k,\uparrow} \\ c_{k,\downarrow} \\ c^\dagger_{-k,\uparrow} \\ c^\dagger_{-k,-\downarrow} \end{pmatrix} \propto \begin{pmatrix} E_k + \xi_k & 0 & d_x(k) - id_y(k) & -d_z(k) \\ 0 & E_k + \xi_k & -d_z(k) & -d_x(k) - id_y(k) \\ -d^*_x(k) - id^*_y(k) & d^*_z(k) & E_k + \xi_k & 0 \\ d^*_z(k) & d^*_x(k) - id^*_y(k) & 0 & E_k + \xi_k \end{pmatrix} \begin{pmatrix} \gamma_{k,\uparrow} \\ \gamma_{k,\downarrow} \\ \gamma^\dagger_{-k,\uparrow} \\ \gamma^\dagger_{-k,-\downarrow} \end{pmatrix}. \tag{18}$$

where I omitted the normalization factor of $1/\sqrt{2E_k(E_k + \xi_k)}$.

Clearly one can dream up some very complicated pairing states for arbitrary choices of $\vec{d}$. We are, however, looking to answer a specific question: *is the peak in the ultrasound attenuation a signature of a sign-changing gap?* To answer this question it will suffice to choose a simple example. I'll consider generic terms in which $\vec{d}$ lies along one of the Cartesian coordinate axes: $\hat{d} = d_\mu(k)\hat{\mu}$ where $\mu \in (x, y, z)$. All of these choices produce unitary gaps, so we will take $d_\mu(k)$ to be a complex-valued function of $\vec{k}$ and decompose it into an amplitude and a phase, $d_\mu(k) = |d_\mu(k)|e^{i\phi(k)}$. In order to proceed with the calculation, we make the same assumption that the interaction is written in the form of Eq. (2) and obeys $M_{k,k'} = \alpha M_{-k',-k}$ where $\alpha = \pm 1$. An additional complication in the case of triplet pairing is that the electron states that are coupled are not necessarily (k, σ) and $(-k, -\sigma)$; for example, picking $\mu = x, y$ couples (k, σ) and $(-k, \sigma)$ instead. This is summarized in the first two columns of Table I. In each case we add the two scattering states that connect the same Bogoliubov quasiparticle states, $c^\dagger_{k,\sigma}c_{k',\sigma} + \alpha\, c^\dagger_{-k',\pm\sigma}c_{-k,\pm\sigma}$, with $\alpha = \pm 1$ again distinguishing Case I and Case II processes (and $\pm\sigma$ is σ for $\mu = z$ and $-\sigma$ for $\mu = x$, y). It's worth noting here that Cases I and II no longer necessarily correspond to parity under time reversal due to the complications associated with spin flipping. By choosing an interaction Hamiltonian that does not flip spins, however, the Case I/II conditions for the s-wave superconductor

$\hat{d}$	Spin Pairing	$\gamma^\dagger\gamma$, $\sigma=\sigma'$	$\gamma\gamma$, $\sigma=\sigma'$	$\gamma^\dagger\gamma$, $\sigma\neq\sigma'$	$\gamma\gamma$, $\sigma\neq\sigma'$
$\hat{x}$	$\lvert\uparrow\uparrow\rangle - \lvert\downarrow\downarrow\rangle$	$-$	$+$	$-$	$+$
$\hat{y}$	$\lvert\uparrow\uparrow\rangle + \lvert\downarrow\downarrow\rangle$	$-$	$-$	$+$	$+$
$\hat{z}$	$\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle$	$-$	$+$	$+$	$-$
singlet	$\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle$	$-$	$+$	$-$	$+$

TABLE I. Signs of the coherence factors for various terms in the p-wave expansion with $\hat{d} = d_\mu(k)\hat{\mu}$. The entries $\pm$ correspond to the different coherence factors in Eq. (19) and *not* to the values of α. The heading $\gamma^\dagger\gamma$ denotes quasiparticle scattering terms and $\gamma\gamma$ denotes creation and annihilation terms; $\sigma \neq \sigma'$ and $\sigma = \sigma$ denote matrix elements with and without a spin flip, respectively (see Appendix A). The results of singlet pairing are included below for comparison.

map directly on to the Case I/II conditions for the p-wave superconductor.

Expanding these terms in the Bogoliubov basis, we find that the coherence factors all take the form

$$F_\mu^\pm(k,k';\alpha) = 1 \pm \alpha \frac{|d_\mu(k)||d_\mu(k')|}{E_k E_{k'}} \cos\big(\phi(k) - \phi(k')\big), \tag{19}$$

which clearly generalizes Eq. (9). The results of the calculation are summarized in Table I, where $+$ denotes the use of $F_\mu^+(k,k';\alpha)$ and $-$ denotes the use of $F_\mu^-(k,k';\alpha)$. Only the first two columns are of use to us as they pertain to matrix elements that do not flip the spin ($\sigma = \sigma'$). The sign conventions of the singlet case are included in the last row for comparison.

Interestingly, we find that choosing $\vec{d}$ to be along the x and z axes produces analogous coherence factors to the singlet case, but $\vec{d}$ along the y axis gives creation and annihilation ($\gamma\gamma$) processes the same coherence factor as scattering ($\gamma^\dagger\gamma$) processes. We can therefore say confidently that the peak in the density of states is not a *signature* of a sign-changing gap, at least not at the level of this analysis. With that said, certain p-wave gaps may disrupt the exact Case I cancellation between the coherence factor and density of states that was seen for s-wave superconductors.

While it may now be tempting to reproduce the calculation for much more complicated triplet pairing states, we can see on the basis of this simple calculation alone that the results will not paint a clear picture. We will therefore ask another somewhat restrained question: *Can this mismatch of coherence factors for $\hat{d} = \hat{y}$ actually produce a peak in the sound attenuation constant?* Given that the coherence factors were not altogether reversed, it is worth asking whether this effect is strong enough to produce a peak. Furthermore, a p-wave gap may have nodes that would decrease the peak in the density of states near the gap edge, further diminishing any potential peak. To answer this question we have to be more specific about the form of $\vec{d}$. I will assume the simple form

$$\vec{d}(k) = \Delta_0 \left(\frac{\hat{z} \cdot \vec{k}}{k_F} \right) \hat{y} \tag{20}$$

where we are assuming all momenta are close to the Fermi surface so $|\vec{k}| = k_F$. This gap has a line of nodes where $k_z = 0$. We now have to perform an angular average within the Fermi's golden rule calculation as the gap is a function of $\hat{z} \cdot \vec{k}$. Analogous to Eq. (10), we have

$$\begin{aligned} \Gamma(\omega, T) = \int_{-\infty}^{\infty} dE_1 \, dE_2 \int \frac{d\Omega_1 d\Omega_2}{(4\pi)^2} |M|^2 N_s(E_1, \Omega_1) N_s(E_2, \Omega_2) \times \\ \Big(F^-(E_1, E_2, \Omega_1, \Omega_2; \alpha) \big(f(E_1) - f(E_2)\big) \delta(E_1 - E_2 - \hbar\omega) \\ + F^-(E_1, E_2, \Omega_1, \Omega_2; \alpha) \big(1 - f(E_1) - f(E_2)\big) \delta(E_1 + E_2 + \hbar\omega) \Big) \end{aligned} \tag{21}$$

where Ω_1 and Ω_2 are the solid angles for the momenta k and k', respectively (the radial component has been transformed into $E_{1/2}$). The density of states is given by

$$N_s(E, \Omega) = N_s(E, \theta, \phi) = N(0) \frac{|E|}{\sqrt{E^2 - \Delta_0^2 \cos^2(\theta)}} \tag{22}$$

and the coherence factors are

$$F^{\pm}(E_1, E_2, \Omega_1, \Omega_2; \alpha) = \frac{1}{2} \left(1 \pm \alpha \frac{\Delta_0^2 |\cos(\theta_1) \cos(\theta_2)|}{E_1 E_2} \right). \tag{23}$$

We now want to integrate over both solid angles. Before doing that, however, we can integrate over E_2 and make the signs of E_1 and E_2 explicit. This is what gave us Eq. (11) from Eq. (10). For the first term of Eq. (21) we take $E_1 > 0$ and $E_2 > 0$, and for the second term we take $E_1 > 0$ and $E_2 < 0$. In the s-wave case these were the only terms that would survive as $\omega \to 0$. In this case, however, the gap has nodes and therefore the density of states does not vanish until $E = 0$. We may therefore have to worry about the pair-breaking

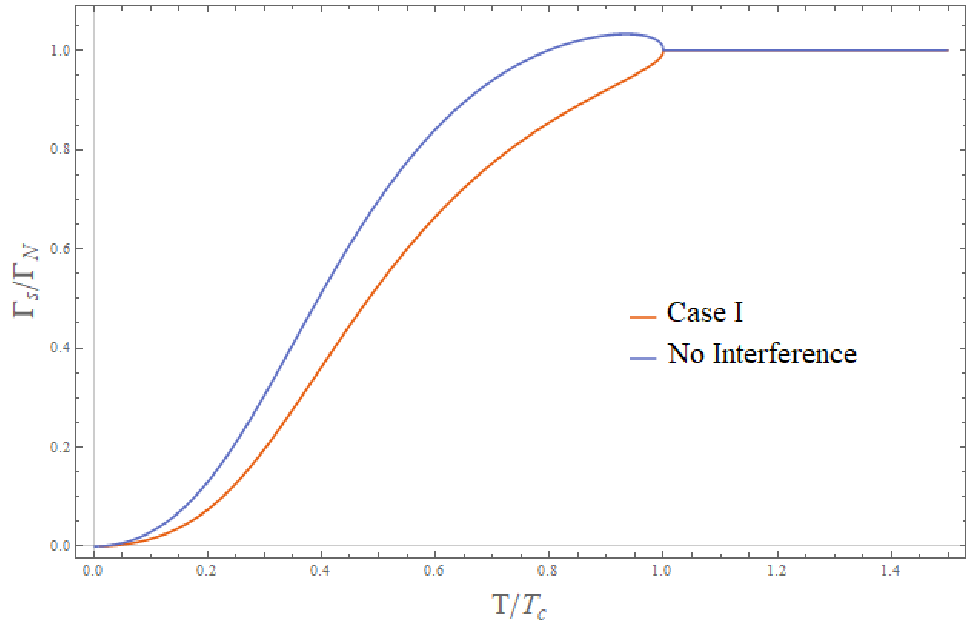

FIG. 2. Plot of the scattering rate for a p_y-gap superconductor ("No Interference") as well as the standard Case I scattering rate, which would be valid for p_x and p_z-gap superconductors.

processes we neglected earlier. I argue that these are irrelevant in Appendix B; here we will proceed to ignore them. Inserting the explicit sign conventions, we have

$$\Gamma(\omega, T) = \int_0^\infty dE \int \frac{d\Omega_1 d\Omega_2}{(4\pi)^2} |M|^2 N_s(E, \Omega_1) N_s(E + \hbar\omega, \Omega_2) \big(f(E) - f(E + \hbar\omega)\big). \quad (24)$$

Interestingly, we find that the coherence factors cancel. With this simplification, we can perform the angular averages over each function $N_s(E, \Omega)$ independently, obtaining the well-known result

$$N_s(E) = \begin{cases} \frac{\pi}{2}(|E|/\Delta_0) & |E| < \Delta_0 \\ (|E|/\Delta_0) \arcsin\big(\Delta_0/|E|\big) & |E| \geq \Delta_0. \end{cases} \quad (25)$$

Expanding in the limit of small ω then gives

$$\Gamma(\omega, T) \approx \beta\omega \int_0^\infty dE \, \big(N_s(E)\big)^2 \frac{e^{\beta E}}{(1 + e^{\beta E})^2} \quad (26)$$

The result, evaluated numerically, is shown in Fig. 2 (again we have divided by Γ_N, which is defined in Eq. (12). Note that there is a peak, although it is significantly less pronounced than the s-wave Hebel-Slichter peak in Fig. 1 or the observed peak in Sr_2RuO_4 [6].

We have shown that a peak in the ultrasound attenuation can arise due to a sign-changing gap, although this is not a signature of sign-changing gaps in general. At this point it's worth stepping back and reviewing the reasonableness of this proposed explanation for the peak in Sr_2RuO_4. This calculation finds an increase in the scattering rate due to the large density of states near the gap. This is a rather spectacular effect in fully-gapped superconductors, which have very large peaks in the density of states as $E \to \Delta$. The presence of nodes in the gap decreases this peak in the density of states, however, which is why the curve in Fig. 2 has a smaller peak than the "No Interference" curve in Fig. 1. While much is uncertain about the structure of the gap in Sr_2RuO_4, there is good evidence that it has nodes [12–16]. Furthermore, p-wave superconductors are not subject to Anderson's theorem [17] and are therefore subject to pair-breaking from non-magnetic impurity scattering. It has been shown that this impurity scattering can strongly renormalize the density of states, broadening peaks and increasing spectral weight at low energies [18]. Indeed, it has already been shown that the resistivity of Sr_2RuO_4 is sensitive to non-magnetic impurity scattering and that the decrease in the effective T_c with the scattering lifetime is in line with the predictions of Abrikosov-Gorkov theory [19]. Collectively, these observations weaken the case for a simple density of states explanation for the peak in sound attenuation of Sr_2RuO_4 as the peak would likely be smaller and significantly less sharp than the experimental observations [6].

IV. ALTERNATIVE EXPLANATION

I will now briefly assess the applicability of an alternative explanation for the peak in ultrasound attenuation. The proposal derives from a numerical investigation by Coffey [7] which sought to explain the peak in the attenuation constants of heavy-fermion superconductors such as UBe_{13} and UPt_3. The method for calculating the ultrasound attenuation was put forth in an earlier paper by Tsuneto [20], although a simpler version applicable to normal metals is derived in Kittel [2]. The essential idea is that a sound wave propagating through a material displaces impurities and creates a long-range electric field due to the motion of the ions. Both of these effects are treated as perturbations. Given that the total electric field in the material is sourced by and gives rise to the electron current, $\vec{j}$ and $\vec{E}$ satisfy a self-consistency equation due to Maxwell's equations in materials. Finally, one can use the fact that the total energy released by the electrons through impurity scattering (in the form of Joule heating, $\vec{j} \cdot \vec{E}$) is equal to the energy lost by the sound wave. This gives us the relation $\alpha = \mathrm{Re}(\vec{j} \cdot \vec{E})/\rho v_s |\vec{u}|^2$, where ρ is the ion density, v_s is the sound velocity and

$|\vec{u}|$ is the magnitude of the velocity field (c.f. Eq. (1)).

Needless to say, this can be a messy calculation in a superconductor. For that reason, I won't go into too many details. In general, it should be understood that the calculation includes the phase space effects discussed earlier as well as disorder-renormalization of the density of states. Furthermore, it is likely that this more sophisticated treatment of impurity and phonon scattering has other sources of temperature dependence beyond the scope of our analysis.

With all that being said, Coffey proposes in a later paper [8] that the origin of the peak is due to the anomalously large effective mass. The explanation he provides is technical and rather unintuitive, but it provides a benchmark to test the applicability of his numerical results to the Sr_2RuO_4 experiment. The role of the large effective mass, per Coffey, is to suppress the Fermi velocity (via the relation $m^* v_F = \hbar k_F$, presuming reasonable values for k_F). Thus, the anomalous ratio Coffey actually identifies is v_s/v_F, the ratio of the sound velocity to the Fermi velocity (this is equal to $\omega\tau/ql$ where $\omega = v_s q$ is the dispersion of the sound wave, l is the mean free path, and τ is the scattering lifetime). His simulations use the ratio $v_s/v_F = 2/3$, whereas a more conventional value would be on the order of 10^{-2}. In Sr_2RuO_4 there are three Fermi surfaces, each with a Fermi velocity of approximately 1×10^{-5} m/s [21]. The sound velocity of each shear and compression mode can be inferred from their respective elastic moduli, C, and the mass density, ρ, via $v = \sqrt{C/\rho}$; from the values in Ref. [22] we find a sound velocity of around 5000 m/s for the modes in question. This gives us a characteristic ratio $v_s/v_F \approx 0.05$, which is reasonably close to the "conventional value" and significantly less than 2/3.

We cannot, however, definitively rule out Coffey's mechanism for the peak. Fig. 3, reproduced from Ref. [7], shows that the ultrasound attenuation of the polar state ($|\vec{d}| \propto k_z/k_F$) with $v_s/v_F = 0.02$ has a peak below T_c. It is hard to discern whether this feature is actually due to the effective mass mechanism he provides, however, as the plot shows strong disorder renormalization: a slow rise below T_c followed by a long plateau down to $T = 0$. This differs significantly from his other plots, which is sensible because the disorder parameter is much larger in Fig. 3. Disorder renormalization of the density of states is controlled by Γ/Δ_0 where Γ is the scattering rate. For the data in question, Coffey chose $\Gamma/\Delta_0 = 2/3$; values of order 1 or larger will strongly renormalize the density of states, broadening any peaks and filling in spectral weight down to $E = 0$ [18]. Indeed, the long plateau of Fig. 3 coupled with

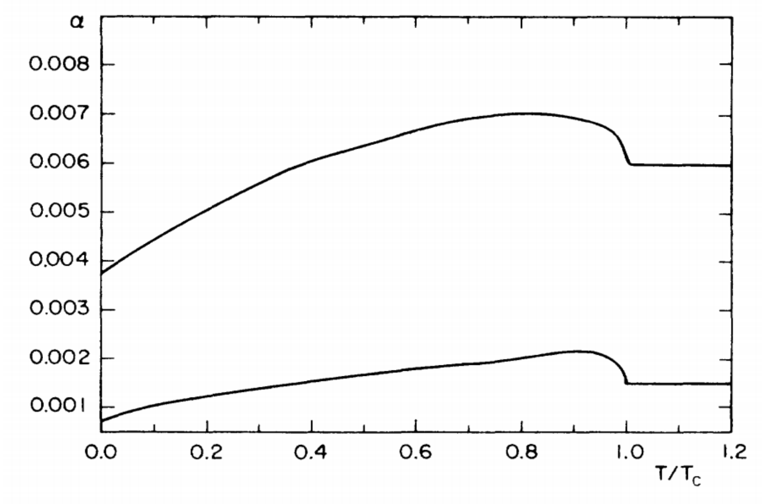

FIG. 3. Ultrasound attenuation as a function of temperature for $v_s/v_F = 0.02$ and $\Gamma/\Delta_0 = 2/3$, reproduced from Fig. 10 of Ref. [8]

a finite ultrasound attenuation at $T = 0$ would indicate that disorder is a significant factor in this result.

Whether the disorder is responsible for the peak itself is less clear. Qualitatively the shape of the peak in Fig. 3 differs significantly from that observed in Ref. [6]; the latter is a sharper, step-like increase. With that said, it is worth noting that the Sr_2RuO_4 experiment *is* in a short-lifetime regime: inferring a BCS gap magnitude of $\Delta_0 \sim 0.2$ meV from T_c [6] and a scattering rate of approximately 1 ps [21] gives a ratio $\Gamma/\Delta_0 \approx 3$. Finally, while the step-like increase is inconsistent with Fig. 3, other strong-disorder plots such as Fig. 9 of Ref. [7] seem to capture the qualitative features of the experimental peak.

V. CONCLUSIONS

Here we investigated the question of whether a peak in the ultrasound attenuation constant below T_c is indicative of a superconductor with a sign-changing gap. After discussion of the s-wave case, we found that the peak is not a feature of all triplet pairing states. With that said, we did determine that $\vec{d} \propto \hat{y}$ exhibits a cancellation of coherence factors and a resulting peak below T_c. We then discussed a few reasons why this peak is small and, given features of the experiment, why this method is unlikely to describe the peak. Finally, we compared experimental parameters to an existing numerical result that predicts a peak in the sound attenuation. While the ratio v_s/v_F assumes conventional values in Sr_2RuO_4, undercutting the relevance of this analysis, we cannot conclusively rule out this mechanism.

Appendix A: Worrying About Spin

If one were instead to consider an interaction Hamiltonian with matrix elements that could flip spins, such as

$$\mathcal{H}_{\text{int}} \sum_{k_1,k_2} \sum_{\sigma_1,\sigma_2} M_{k_1,\sigma_1;k_2,\sigma_2} c^\dagger_{k_1,\sigma_1} c_{k_2,\sigma_2}, \tag{A1}$$

the situation gets a bit more complicated. I relegated this discussion to an appendix because it adds unnecessary complication to the interactions relevant for ultrasound attenuation.

First consider the s-wave gap. I'll use the notation for the generalized BCS theory (encompassing non-s-wave gaps) throughout. In that language, spin singlet gaps are defined as $\hat{\Delta}_k = i\psi(k)\hat{\sigma}_y$ and an isotropic s-wave gap has $\psi(k) = \Delta$, a constant. The Bogoliubov transformation diagonalizing the mean-field Hamiltonian is

$$\begin{pmatrix} c_{k,\uparrow} \\ c_{k,\downarrow} \\ c^\dagger_{-k,\uparrow} \\ c^\dagger_{-k,-\downarrow} \end{pmatrix} \propto \begin{pmatrix} E_k + \xi_k & 0 & 0 & -\Delta \\ 0 & E_k + \xi_k & \Delta & 0 \\ 0 & -\Delta & E_k + \xi_k & 0 \\ \Delta & 0 & 0 & E_k + \xi_k \end{pmatrix} \begin{pmatrix} \gamma_{k,\uparrow} \\ \gamma_{k,\downarrow} \\ \gamma^\dagger_{-k,\uparrow} \\ \gamma^\dagger_{-k,-\downarrow} \end{pmatrix}. \tag{A2}$$

where I again omit the normalization factor of $1/\sqrt{2E_k(E_k+\xi_k)}$. In order for this phase space argument to be sensible, we must have

$$M_{k_1,\sigma_1;k_2,\sigma_2} = \alpha\ M_{-k_2,-\sigma_2;-k_1,-\sigma_1} \tag{A3}$$

where again $\alpha = \pm 1$ differentiates Cases I and II. This condition means that the interaction is even or odd under time reversal, as mentioned in the main text. As a check we reproduce the calculation performed previously for $\sigma_1 = \sigma_2$, finding

$$\begin{aligned} c^\dagger_{k_1,\uparrow} c_{k_2,\uparrow} + \alpha\ c^\dagger_{-k_2,\downarrow} c_{-k_1,\downarrow} = \frac{1}{2}\left(1 - \alpha\frac{\Delta^2}{E_1 E_2}\right)(\gamma^\dagger_{k_1,\uparrow}\gamma_{k_2,\uparrow} + \alpha\ \gamma^\dagger_{-k_2,\downarrow}\gamma_{-k_1,\downarrow}) \\ + \frac{1}{2}\left(1 + \alpha\frac{\Delta^2}{E_1 E_2}\right)(\gamma^\dagger_{k_1,\uparrow}\gamma^\dagger_{-k_2,\downarrow} + \alpha\ \gamma_{-k_1,\downarrow}\gamma_{k_2,\uparrow}) \end{aligned} \tag{A4}$$

where I simplified the coherence factors as before. Note, however, that the situation differs when one considers a spin-flipping interaction term:

$$c^\dagger_{k_1,\uparrow}c_{k_2,\downarrow} + \alpha\ c^\dagger_{-k_2,\uparrow}c_{-k_1,\downarrow} = \frac{1}{2}\left(1+\alpha\frac{\Delta^2}{E_1E_2}\right)(\gamma^\dagger_{k_1,\uparrow}\gamma_{k_2,\downarrow} + \alpha\ \gamma^\dagger_{k_2,\uparrow}\gamma_{-k_1,\downarrow}) \\ + \frac{1}{2}\left(1-\alpha\frac{\Delta^2}{E_1E_2}\right)(\gamma^\dagger_{k_1,\uparrow}\gamma^\dagger_{-k_2,\uparrow} + \alpha\ \gamma_{-k_1,\downarrow}\gamma_{k_2,\downarrow}). \quad \text{(A5)}$$

The coherence factors are reversed! The problem, addressed in Schrieffer [23], is that we also should have inverted the α signs when considering a spin-flip process. For example, consider the interaction Hamiltonian for NMR:

$$\mathcal{H}_{\text{NMR}} = A\sum_{k,k'} a^*_k a_{k'}\left(I_z(c^\dagger_{k,\uparrow}c_{k',\uparrow} - c^\dagger_{k,\downarrow}c_{k',\downarrow}) + I_+c^\dagger_{k,\downarrow}c_{k',\uparrow} + I_-c^\dagger_{k,\uparrow}c_{k',\downarrow}\right) \quad \text{(A6)}$$

where the product $a^*_{k'}a_k$ comes from the Bloch functions (and is complex conjugated under $k \to -k$) and I is the nuclear magnetic spin. The key here is that time reversal takes

$$\begin{aligned} c^\dagger_{k\uparrow}c_{k',\uparrow} &\to c^\dagger_{-k',\downarrow}c_{-k,\downarrow} \\ c^\dagger_{k,\uparrow}c_{k',\downarrow} &\to c^\dagger_{-k',\uparrow}c_{-k,\downarrow} \end{aligned} \quad \text{(A7)}$$

For the first term, time reversal connects scattering between two spin-up particles and scattering between two spin-down particles. As these feel interactions of the opposite sign, we would subtract them to get the coherence factors. In the second term, however, we see that the σ^+ operation (creating a spin up and destroying a spin down) maps to another σ^+ under time reversal. Thus these terms would add for this Case II process. Conversely, for a Case I process we would add matrix elements that don't flip a spin and subtract those that do. We would therefore find, as stated in the main text, that the coherence factor for scattering processes is $(1-\alpha\Delta^2/E_1E_2)/2$ and the coherence factor for creation/annihilation processes is $(1+\alpha\Delta^2/E_1E_2)/2$ [9, 23].

For the p-wave case, we must address the additional complication that the definitions of Cases I and II differ depending on the particular triplet spin pairing state. This is the primary reason for ignoring them in the main text. The definitions of α for each cardinal direction of $\vec{d}$ are summarized in Table II. This is a necessary accompaniment to the two rightmost columns of Table I as the definitions of α are implicit in those. For $\hat{d} = \hat{z}$, the Bogoliubov transformation pairs the same two electronic states as the singlet Bogoliubov

$\hat{d}$	Spin Pairing	Condition
$\hat{x}$	$\|\uparrow\uparrow\rangle - \|\downarrow\downarrow\rangle$	$M_{k,\sigma;k',\sigma'} = \alpha\ M_{-k',\sigma';-k,\sigma}$
$\hat{y}$	$\|\uparrow\uparrow\rangle + \|\downarrow\downarrow\rangle$	$M_{k,\sigma;k',\sigma'} = \alpha\ M_{-k',\sigma';-k,\sigma}$
$\hat{z}$	$\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle$	$M_{k,\sigma;k',\sigma'} = \alpha\ M_{-k',-\sigma';-k,-\sigma}$
singlet	$\|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle$	$M_{k,\sigma;k',\sigma'} = \alpha\ M_{-k',-\sigma';-k,-\sigma}$

TABLE II. Summary of the definitions of α for different triplet pairing states, which sets the condition for Case I ($\alpha = 1$) and Case II ($\alpha = -1$) processes. Also shown is the pairing in spin space.

transformation, so the condition is identical to the singlet case. For $\hat{d} = \hat{x}$ or $\hat{y}$, however, the Bogoliubov transformation couples states (k, σ) and $(-k, \sigma)$. This means the definition of Case I and II processes must differ in these cases. Choosing $\vec{d}$ not along a particular axis would complicate things even further. We will ignore more complicated configurations because, in the end, the phase space argument made above would only really apply in the case of spin-independent interactions [24]. We can see this by considering the NMR Hamiltonian for a superconducting gap with $\hat{d} = \hat{y}$. For this argument to apply, we would want the matrix elements $M_{k,\sigma;k',\sigma'}$ and $M_{-k',\sigma';-k,\sigma}$ to differ at most by a sign. This is clearly the case when $\sigma = \sigma'$. Considering that term alone would indicate that we might have an $\alpha = 1$ (Case I) process, which would be a simple reversal of the s-wave sign convention. The problem is that the two spin flip terms that are coupled by the Bogoliubov transformation, $c^{\dagger}_{k,\uparrow} c_{k',\downarrow}$ and $c^{\dagger}_{-k',\downarrow} c_{-k,\uparrow}$, would couple to I_- and I_+, respectively, so their associated matrix elements would generically have different magnitudes. The phase space argument would therefore not apply at all to NMR if $\hat{d} = \hat{y}$ (or, more generally, if $\vec{d}$ has any component in the x-y plane). We are fortunate that the problem at hand, (assumed to be) mediated by scattering processes that do not flip the electron spin, can be explained rather cleanly by ignoring all σ dependence in the matrix elements. In that case, the definitions of α can be considered to be the same for generic $\vec{d}$ and for the singlet case, as discussed in the main text.

Appendix B: Pair Breaking

One available explanation for the peak behavior is pair breaking, which corresponds to the processes neglected in Eq. (11). Specifically these are scattering processes (constrained by $\delta(E_1 - E_2 + \hbar\omega)$) where $\text{sign}(E_1) = -\text{sign}(E_2)$ and creation/annihilation processes (constrained by $\delta(E_1 + E_2 - \hbar\omega)$) where $\text{sign}(E_1) = \text{sign}(E_2)$. The reason for neglecting them in the s-wave superconductor is that they only exist for $|\hbar\omega| > 2\Delta$ as the density of states vanishes for $E < \Delta$. A superconductor with nodes, however, has a finite density of states down to $E = 0$ (albeit in particular directions). Thus there should be some amplitude for pair breaking no matter the size of $\Delta(T)$, which one might think would produce a peak in the ultrasound attenuation on its own. Implicit in this discussion is the recognition that these processes would have coherence factors of the opposite sign in the s-wave case (and, more importantly, for the p_x and p_z gaps), thereby producing Hebel-Slichter-like peaks even for Case I interactions.

The purpose of this section is to assuage those fears. I will compute the scattering rate Γ_s/Γ_N for a p-wave superconductor where $\vec{d} \propto \vec{x}$ and $|\vec{d}| = \Delta_0 k_z/k_F$. This gap is a particularly simple choice but it is also illustrative – it has a line node on the surface $k_z = 0$, which means the density of states has substantial support for $|E| < \Delta_0$. We will again take our integral over only positive energies and insert the signs explicitly. Note that the sign restrictions for these processes, in combination with the delta function constraints, implies that $|\hbar\omega| = |E_2| + |E_1|$. Thus, after evaluating the delta function, the final integral will have bounds from $[0, \omega)$. Collecting terms, we have

$$\Gamma^{(2)}(\omega, T) \approx \int_0^\omega dE \int d\Omega_1 d\Omega_2 \; N_s(E, \Omega_1) N_s(E + \hbar\omega, \Omega_2) |M|^2 F^+(E, E + \hbar\omega, \Omega_1, \Omega_2; \alpha) \times \big(f(E)(1 - f(E + \hbar\omega)) - f(E + \hbar\omega)(1 - f(E))\big). \quad \text{(B1)}$$

I've included angular integrals over the solid angle $\Omega_{1/2}$ to account for the fact that the gap is a function of the direction of $\vec{k}$. As before, we have $|\vec{d}(k)| = \Delta_0 \cos(\theta)$ where $\theta \in [0, \pi)$ is the polar angle. Note that this integral vanishes as $\omega \to 0$: pair breaking should be understood as a finite-ω effect. I evaluate this integral numerically for fixed ω and add it to the integral in Eq. (21) (taking the second coherence factor to be F^+ to account for the fact that it is a p_x gap). The result, divided by the result in the normal state, is plotted in Fig. 4. As one might expect, the peak is small, sharp, and only occurs when $\Delta(T) \sim \omega$. I've chosen two

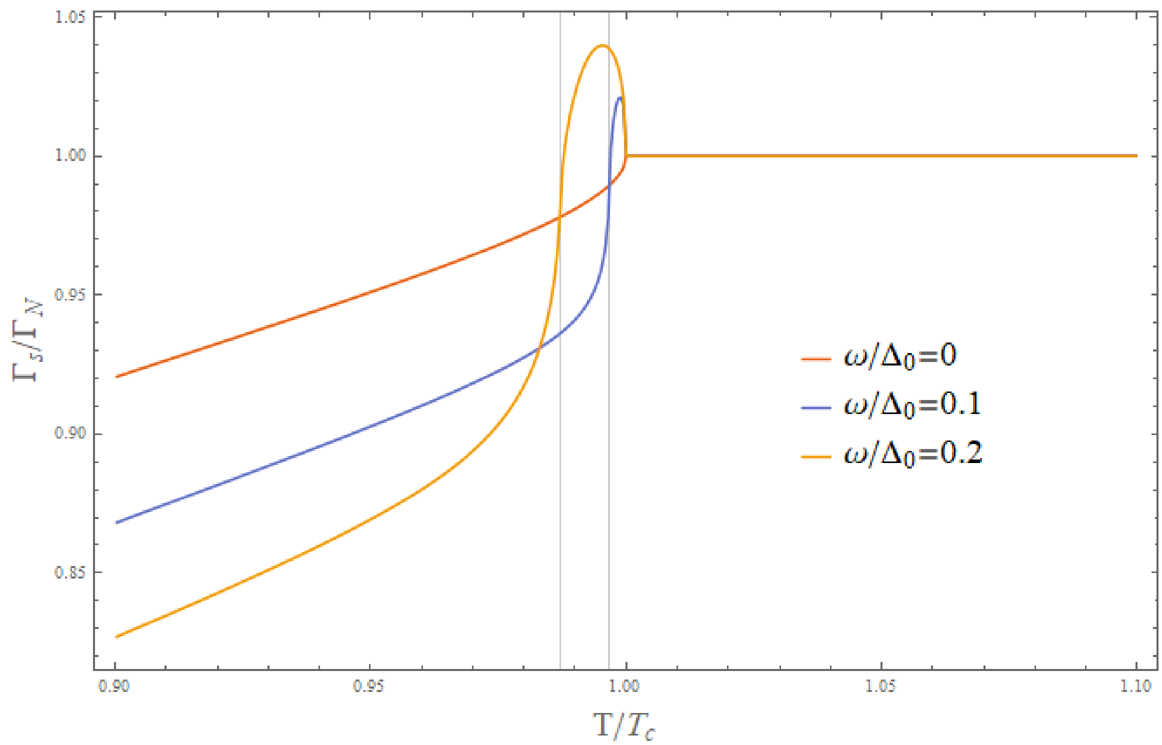

FIG. 4. Plot of the Case I superconducting scattering rate over the normal scattering rate versus temperature for $\vec{d}(k) = \Delta_0 k_z \hat{z}/k_F$. The different curves are for different frequencies, ω, and we have included pair-breaking processes in the calculation. Grid lines show $\Delta(T) = \omega$ for the two finite frequencies. We use the BCS s-wave form of the temperature dependence, $\Delta(T) = \Delta_0\sqrt{1 - T/T_c}$.

values that allow us to visualize the peak and included vertical lines at the points $\Delta(T) = \omega$. In Ref. [6] the authors estimate this transition point using $\Delta_0 = 0.2$ meV and their applied frequency of 2 MHz, finding that the crossover for a BCS temperature dependence would be on the order of a nanokelvin below T_c. This is consistent with Figure 4 and demonstrates that pair breaking, even with a finite density of states down to $E = 0$, is not sufficient to explain the observed peak. Furthermore, given that the effect vanishes in the $\omega \to 0$ limit, we are justified in ignoring such terms for conventional applications.

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[24] *Only* is clearly too restrictive here. Scenarios can be envisioned in which the phase space argument might give interesting results. I don't explore this further, however, because my assumption is that generic interaction Hamiltonians will be more complicated and will therefore yield neither the peaks of Hebel and Slichter nor the cliffs of pure destructive interference.